Algebraic geometry approach to the Bethe equation for Hofstadter-type models

2002 J. Phys. A: Math. Gen. 35 5907

(http://iopscience.iop.org/0305-4470/35/28/310)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 5907–5933 PII: S0305-4470(02)33094-4

Algebraic geometry approach to the Bethe equation for Hofstadter-type models

Shao-Shiung Lin[1] **and Shi-Shyr Roan**[2,3]

[1] Department of Mathematics, Taiwan University, Taipei, Taiwan
[2] Institute of Mathematics, Academia Sinica, Taipei, Taiwan

E-mail: lin@math.ntu.edu.tw and maroan@ccvax.sinica.edu.tw

Received 28 January 2002, in final form 9 May 2002
Published 5 July 2002
Online at stacks.iop.org/JPhysA/35/5907

Abstract

We study the diagonalization problem of certain Hofstadter-type models through the algebraic Bethe ansatz equation by the algebraic geometry method. When the spectral variables lie on a rational curve, we obtain the complete and explicit solutions for models with a rational magnetic flux, and discuss the Bethe equation of their thermodynamic flux limit. The algebraic geometry properties of the Bethe equation on high genus algebraic curves are investigated in accordance with physical considerations of the Hofstadter model.

PACS numbers: 02.10.Rn, 03.65.Fd, 5.30, 75.10.Jm

1. Introduction

The Hofstadter Hamiltonian is defined by

$$H_{\mathrm{Hof}} = \mu(\alpha U + \alpha^{-1}U^{-1}) + \nu(\beta V + \beta^{-1}V^{-1}) \tag{1}$$

where U, V are unitary operators satisfying the Weyl commutation relation for an absolute value 1 complex number ω, $UV = \omega VU$, and α, β, μ, ν are parameters with $|\alpha| = |\beta| = 1$, $\mu, \nu \in \mathbf{R}$. For a primitive Nth root of unity $\omega\ (= \mathrm{e}^{2\pi\sqrt{-1}\Phi})$, i.e. the phase factor $\Phi = P/N$ with P relatively prime to N, one may assume the Nth power identity of $U, V : U^N = V^N = 1$. There are several physical interpretations for the Hofstadter model, especially in solid state physics. A prominent one is to consider it as a tight-binding approximation for electrons bound to atomic sites in a two-dimensional crystal and in a strong external magnetic field. The history of the Hofstadter model can be traced back to the work of Peierls [20] on Bloch electrons in metals in the presence of a constant external magnetic field. By the pioneering works in the 1950s and 1960s [1, 10, 14, 19, 25], the role of magnetic translations was found

[3] Supported in part by the NSC grant of Taiwan.

for the Hamiltonian (1), where the Weyl pair of operators, αU and βV, is a discrete version of magnetic translations in the x, y directions, with the phase Φ of their commutation factor ω representing the magnetic flux (per plaquette), and μ, ν are the hopping amplitudes of the system. Subsequently, a systematic study of this 2D lattice model had begun. With the discovery of quantum Hall effects, a large number of interesting and important theoretical papers on Hamiltonians of the Hofstadter type appeared in the 1980s for the quantum mechanical interpretation of the Hall conductivity plateaus (see, e.g., [13] and references therein). In 1976, Hofstadter [16] found the butterfly figure of the spectral band versus the magnetic flux, in which a beautiful fractal picture is exhibited. A detailed study of the model and other Hofstadter-type models began thereafter on the explanation of the fractal structure through various mathematical approaches, such as the semiclassical approximation, WKB-analysis on the difference equation, non-commutative geometry and others [2, 7, 8, 11, 15, 24]. A pedagogical account of this important aspect can be found in a vast literature (e.g. [6, 21] and references therein).

On the other hand, motivated by the work of Wiegmann and Zabrodin [23] on the appearance of quantum $U_q(sl_2)$ symmetry in problems of magnetic translation, Faddeev and Kashaev [12] pursued the diagonalization problem on the following type of Hamiltonian by the quantum transfer matrix method developed by the Leningrad school in the early 1980s (see, for instance, [22]):

$$H_{\mathrm{FK}} = \mu(\alpha U + \alpha^{-1}U^{-1}) + \nu(\beta V + \beta^{-1}V^{-1}) + \rho(\gamma W + \gamma^{-1}W^{-1})$$

where U, V, W are unitary operators with the Weyl commutation relations and the Nth power identity property, $UV = \omega VU$, $VW = \omega WV$, $WU = \omega UW$; $U^N = V^N = W^N = 1$. With $\rho = 0$ in H_{FK}, one has the Hofstadter Hamiltonian (1) with a rational flux. Though the physical content of the extended Hamiltonian H_{FK} with one more added operator W has not yet been clarified, this general formulation will provide a nice setting for the study of this kind of Hofstadter-type Hamiltonian by the quantum inverse scattering method. With the solution of the Bethe ansatz equation under a certain postulated degree condition (see (5.26) of [12]), the energy expression of (1) obtained in [23] was reproduced in the $\rho = 0$ limit computation. Furthermore, a general framework for calculating spectra of the Hamiltonian through the six-vertex model by the algebraic Bethe ansatz method was presented in [12]. In this approach, the Bethe ansatz equation is formulated through the Baxter vector[4] [3, 5], visualized on a 'spectral' curve associated with the corresponding Hofstadter-type model. In general, the spectral curve is a Riemann surface with a very high genus. Thus here the Bethe equation can be viewed as a version of Baxter's T-Q relation [3] on a high genus spectral curve. The method relies on a special local monodromy solution of the Yang–Baxter equation for the six-vertex R-matrix. This solution also appeared in the study of chiral Potts model [4]. For a finite size L, the trace of the L-monodromy matrix gives rise to the transfer matrix acting on the quantum space $\overset{L}{\otimes}\, \mathbf{C}^N$, where N is the order of the rational flux. The transfer matrix is composed of a set of commuting operators, one of which is a Hofstadter-like Hamiltonian. For the physical consideration, the size L is kept to be a fixed number. While motivated by the Hofstadter butterfly spectral figure, the 'thermodynamic' limit will be treated in the manner of the rational flux's order N tending to infinity. This process is a different kind of limiting procedure from many in quantum integrable systems, such as the *XYZ*-spin chain or its degenerated forms. For the study of models in this paper, we are mainly concerned with the problem of diagonalizing the transfer matrix of size $L = 3$ (see [12], or sections 6 and 7 of the paper). The Bethe ansatz equations of the Hofstadter-like Hamiltonians

[4] It is also called 'Baxter vacuum state' in other literature. Here we follow the terminology used in [12].

arising in such manner were clearly proposed in [12], and the principle could be equally applied to more general, possibly interesting, models. However, the explicit solutions and their qualitative nature are not yet known, even in the situation when the spectral curve is a rational one. The aim of this paper is to show that the algebraic geometry method could provide an effective tool for a thorough investigation into the mathematical structure of the Bethe equation along the line of the quantum transfer matrix scheme in [12]. In this work, we obtain the complete solutions of the Bethe equation for models with a rational spectral curve for $L \leqslant 3$, among which a special kind of Hofstadter-type Hamiltonian $H_{\rm FK}$ is treated. We present a detailed and rigorous mathematical derivation of these solutions of Bethe ansatz equations, including that in [12], and further expand it to all the other sectors. Both the qualitative and quantitative properties of the solutions are studied thoroughly. With the understanding of Bethe solutions for these Hofstadter-like models of the rational flux, we apply the thermodynamic process which enables us to propose the Bethe equation for a generic flux with the desirable solutions from both mathematical and physical considerations. The analysis we make on the solution of the model via the Bethe ansatz method has clearly revealed the algebraic geometry character of the Bethe equation. We adopt such an interpretation in approaching the diagonalization problem of certain quantum integrable models. Accordingly, the Bethe ansatz method for the Hofstadter model (1) is formulated along this scheme, and the qualitative nature of the Bethe solutions is obtained through the approach of algebraic curve theory.

This paper is organized as follows. In section 2, we outline the concept of the Baxter vector, a key ingredient for the study of the Bethe (ansatz) equation throughout this work. In order to gain conceptual clarity, we shall present the subject from the algebraic geometry aspect. In section 3, we first recall some results in [12] relevant to our further discussion, fix notations on the transfer matrix and Bethe equation, then derive some general qualitative properties about eigenvalues of the transform matrix. In the next four sections, we shall consider the case where the spectral data lie on a rational curve, and perform the mathematical derivation of the answer, along with the discussion of their physical applications. In section 4, we briefly review the basic procedure of limit reduction to a rational spectral curve, that originally appeared in [12], and present an explicit form of Baxter vector, which will be used in further discussions. We present the complete solutions of the Bethe equations of all sectors for $L \leqslant 3$ in section 5. In section 6, we explain the 'degeneracy' relations between the Bethe solutions and the eigenspaces in the quantum space of the transfer matrix. We also mention the relationship between our Bethe solutions and some known results by using the Bethe ansatz method in the literature, and indicate the difference of our approach to the Bethe equation with that of the usual Bethe-ansatz-type technique through certain non-physical solutions obtained by the latter method. In section 7, we will address the 'thermodynamic' limit problem in the sense that N tends to ∞, and discuss the Bethe equation for those Hofstadter-like Hamiltonians that appeared previously in the context of section 5, but with a generic q. In section 8, we study the Hofstadter Hamiltonian (1) by starting from the format in [12]. Then we go through an algebraic geometry analysis of the high genus spectral curve in which solutions of the Bethe equation are represented, and the connection of the spectral curves to elliptic curves is then clarified. We obtain a primary understanding of Bethe solutions through the spectral curve and their relation with the Heisenberg algebra. We end with the concluding remarks in section 9 with a discussion of the directions of our further enquiry.

Convention: in this paper, $\mathbf{R}$, $\mathbf{C}$ will denote the field of real, complex numbers respectively, and $\mathbf{Z}$ the ring of integers. For positive integers N, n, we will denote by $\overset{n}{\otimes}\, \mathbf{C}^N$ the tensor product of n-copies of the complex N-dimensional vector space $\mathbf{C}^N$, and by $\mathbf{Z}_N$ the quotient ring $\mathbf{Z}/N\mathbf{Z}$.

2. Algebraic geometry preliminary and Baxter vector

In this paper, we shall denote by ω a primitive Nth root of unity, and $q = \omega^{\frac{1}{2}}$ with $q^N = (-1)^{N+1}$, in particular for odd $N, q = \omega^{\frac{N+1}{2}}$. Let Z, X be two operators satisfying the Weyl commutation relation with the Nth power identity,

$$ZX = \omega XZ \qquad Z^N = X^N = I.$$

The algebra generated by Z, X is called the Weyl algebra, in which the element ZX will be denoted by $Y := ZX$. The following relations hold:

$$XY = \omega^{-1}YX \qquad YZ = \omega^{-1}ZY \qquad Y^N = (-1)^{N-1}.$$

The canonical irreducible representation of the Weyl algebra is given by the following expressions on the N-dimensional space $\mathbf{C}^N$:

$$\begin{array}{ll} Z : v \mapsto Z(v) & Z(v)_k = q^{2k}v_k \\ X : v \mapsto X(v) & X(v)_k = v_{k-1} \\ Y : v \mapsto Y(v) & Y(v)_k = q^{2k}v_{k-1}. \end{array} \tag{2}$$

Here a vector v of $\mathbf{C}^N$ is represented by a sequence of coordinates, $v = (v_k)_{k\in\mathbf{Z}}$, with the N-periodic condition, $v_k = v_{k+N}$. Hence one can consider the index k as an element of $\mathbf{Z}_N$ in what follows, if no confusion arises. Denote by $|k\rangle$ the standard basis of $\mathbf{C}^N$, by $\langle k|$ the dual basis of $\mathbf{C}^{N*}$ for $k \in \mathbf{Z}_N$. The kth component of a vector v of $\mathbf{C}^N$ is given by $v_k = \langle k|v\rangle$. In the Weyl algebra, we shall consider only the vector subspace spanned by X, Y, Z and the identity I, in which we denote the non-zero operator as

$$\varphi_{\alpha,\beta,\gamma,\delta} := \alpha Y - \beta X - \gamma Z + \delta I : \mathbf{C}^N \longrightarrow \mathbf{C}^N.$$

The kernel of the above operator, $\mathrm{Ker}(\varphi_{\alpha,\beta,\gamma,\delta})$, depends only on the ratio of the coefficients, i.e. the element $[\alpha, \beta, \gamma, \delta]$ in the projective 3-space $\mathbf{P}^3$. The non-triviality of $\mathrm{Ker}(\varphi_{\alpha,\beta,\gamma,\delta})$ defines a hypersurface of $\mathbf{P}^3$,

$$\mathcal{F} := \{[\alpha, \beta, \gamma, \delta] \in \mathbf{P}^3 \mid \mathrm{Ker}(\varphi_{\alpha,\beta,\gamma,\delta}) \neq 0\}.$$

In fact, one has the defining equation of $\mathcal{F}$ as follows:

Lemma 1. *The surface $\mathcal{F}$ is defined by the equation*

$$\mathcal{F} : \alpha^N + \delta^N = \beta^N + \gamma^N \qquad [\alpha, \beta, \gamma, \delta] \in \mathbf{P}^3.$$

Furthermore for $[\alpha, \beta, \gamma, \delta] \in \mathcal{F}, \mathrm{Ker}(\varphi_{\alpha,\beta,\gamma,\delta})$ is a one-dimensional subspace of $\mathbf{C}^N$ generated by a vector $v = (v_m)$ with the ratios $v_m : v_{m-1} = (\alpha\omega^m - \beta) : (\gamma\omega^m - \delta)$.

Proof. For $v \in \mathbf{C}^N$, it is obvious that the criterion of v in $\mathrm{Ker}(\varphi_{\alpha,\beta,\gamma,\delta})$ is described by the above ratio relations of v_m. For a non-zero vector v of $\mathbf{C}^N$, the periodic relation $v_m = v_{m+N}$ for a non-zero component v_m gives rise to the equation of $\mathcal{F}$. □

Let v be a basis of $\mathrm{Ker}(\varphi_{\alpha,\beta,\gamma,\delta})$ for $[\alpha, \beta, \gamma, \delta] \in \mathcal{F}$. Note that there are N^2 (projective) lines in $\mathcal{F}$, defined by $\alpha^N - \beta^N = 0$, equivalently, $\gamma^N - \delta^N = 0$, which are labelled by $\mathbf{P}^1_{j,k} := \{\alpha\omega^j - \beta = \gamma - \delta\omega^k = 0\}$ for $j, k \in \mathbf{Z}_N$. Outside these lines, the components of v are all non-zero. For elements in $\mathbf{P}^1_{j,k}$, one can set $v = |k\rangle$ at $[0, 0, \omega^k, 1]$, and $v = |j-1\rangle$ at $[1, \omega^j, 0, 0]$; while for the rest of the elements, the indices in $\mathbf{Z}_N$ with the non-zero components of v form a chain from k increasing to $j-1$.

For future purposes, we introduce a family of non-homogenous representations of the surface $\mathcal{F}$, depending on the parameter $h = [a, b, c, d] \in \mathbf{P}^3$,

$$\alpha = \xi' a \qquad \beta = xb \qquad \gamma = -\xi'\xi xc \qquad \delta = -\xi d$$

where $(x, \xi, \xi') \in \mathbf{C}^3$ satisfies the equation

$$\mathcal{S}_h : \xi'^N a^N - x^N b^N = (-\xi)^N (x^N \xi'^N c^N - d^N).$$

For a fixed $h \in \mathbf{P}^3$, the operator $\varphi_{\alpha,\beta,\gamma,\delta}$ corresponding to (x, ξ, ξ') becomes

$$F(x, \xi, \xi')(= F(x, \xi, \xi'; h)) := \xi' aY - xbX + \xi'\xi xcZ - \xi dI \tag{3}$$

and we have

$$\begin{aligned} F(x, \xi - 1, \xi') &= F(x, \xi, \xi') - \xi' xcZ + dI \\ F(x, \xi, \xi' - 1) &= F(x, \xi, \xi') - \xi xcZ - aY. \end{aligned}$$

For an element $p = (x, \xi, \xi')$ of $\mathcal{S}_h$, we shall denote $|p\rangle$ as the basis of $\mathrm{Ker}(\varphi_{\xi' a, xb, -\xi'\xi xc, -\xi d})$ with $\langle 0|p\rangle = 1$, equivalently, $|p\rangle$ is the vector of $\mathbf{C}^N$ defined by

$$\langle 0|p\rangle = 1 \qquad \frac{\langle m|p\rangle}{\langle m-1|p\rangle} = \frac{\xi' a\omega^m - xb}{-\xi(\xi' xc\omega^m - d)}.$$

We shall call $|p\rangle$ the Baxter vector associated with $p \in \mathcal{S}_h$ [3, 5, 12]. Then

$$\begin{aligned} F(x, \xi, \xi')|p\rangle &= \vec{0} \\ F(x, \xi - 1, \xi'; h)|p\rangle &= |\tau_- p\rangle \Delta_-(p) \\ F(x, \xi, \xi' - 1, ; h)|p\rangle &= -|\tau_+ p\rangle \Delta_+(p) \end{aligned} \tag{4}$$

where $\Delta_\pm$ are the following (rational) functions of $\mathcal{S}_h$:

$$\Delta_-(x, \xi, \xi') = d - x\xi' c \qquad \Delta_+(x, \xi, \xi') = \frac{\xi(ad - x^2 bc)}{\xi' a - xb}$$

and $\tau_\pm$ are the automorphisms of $\mathcal{S}_h$ defined by $\tau_\pm(x, \xi, \xi') = (q^{\pm 1} x, q^{-1}\xi, q^{-1}\xi')$.

3. The transfer matrix and the Bethe equation

As in [12], it is known that the following L-operator[5] with the operator-valued entries acting on the quantum space $\mathbf{C}^N$,

$$L_h(x) = \begin{pmatrix} aY & xbX \\ xcZ & d \end{pmatrix} \qquad x \in \mathbf{C}$$

possesses the intertwining property of the Yang–Baxter relation,

$$R(x/x')(L_h(x) \bigotimes_{\text{aux}} 1)(1 \bigotimes_{\text{aux}} L_h(x')) = (1 \bigotimes_{\text{aux}} L_h(x'))(L_h(x) \bigotimes_{\text{aux}} 1) R(x/x') \tag{5}$$

where 'aux' will always indicate an operation taking on the auxiliary space $\mathbf{C}^2$, $R(x)$ is the matrix of a 2-tensor of the auxiliary space with the following numerical expression:

$$R(x) = \begin{pmatrix} x\omega - x^{-1} & 0 & 0 & 0 \\ 0 & \omega(x - x^{-1}) & \omega - 1 & 0 \\ 0 & \omega - 1 & x - x^{-1} & 0 \\ 0 & 0 & 0 & x\omega - x^{-1} \end{pmatrix}.$$

By performing the matrix product on auxiliary spaces and the tensor product of quantum spaces, one has the L-operator associated with an element $\vec{h} = (h_0, \ldots, h_{L-1}) \in (\mathbf{P}^3)^L$,

$$L_{\vec{h}}(x) = \bigotimes_{j=0}^{L-1} L_{h_j}(x) := L_{h_0}(x) \otimes L_{h_1}(x) \otimes \cdots \otimes L_{h_{L-1}}(x) \tag{6}$$

[5] The conventions in this paper are different from those used in [12] where X, Y correspond to our Z, X here.

which again satisfies relation (5). The entries of $L_{\vec{h}}(x)$ are operators of the quantum space $\overset{L}{\otimes}\, \mathbf{C}^N$, and its trace defines the transfer matrix

$$T_{\vec{h}}(x) = \mathrm{Tr}_{\mathrm{aux}}(L_{\vec{h}}(x)).$$

Then the commutation relation holds,

$$[T_{\vec{h}}(x), T_{\vec{h}}(x')] = 0 \qquad x, x' \in \mathbf{C}.$$

The transfer matrix $T_{\vec{h}}(x)$ can also be computed by changing L_{h_j} to $\widetilde{L}_{h_j}$ via a gauge transformation in the following manner:

$$\widetilde{L}_{h_j}(x) = A_j L_{h_j}(x) A_{j+1}^{-1} \qquad A_L := A_0 \qquad 0 \leqslant j \leqslant L-1. \tag{7}$$

Set

$$A_j = \begin{pmatrix} 1 & \xi_j - 1 \\ 1 & \xi_j \end{pmatrix}$$

and denote the corresponding $\widetilde{L}_{h_j}(x)$ by $\widetilde{L}_{h_j}(x, \xi_j, \xi_{j+1})$. With $F_h(x, \xi, \xi')$ of (3), we have

$$\widetilde{L}_{h_j}(x, \xi_j, \xi_{j+1}) = \begin{pmatrix} F_{h_j}(x, \xi_j - 1, \xi_{j+1}) & -F_{h_j}(x, \xi_j - 1, \xi_{j+1} - 1) \\ F_{h_j}(x, \xi_j, \xi_{j+1}) & -F_{h_j}(x, \xi_j, \xi_{j+1} - 1) \end{pmatrix}$$

and

$$T_{\vec{h}}(x) = \mathrm{Tr}_{\mathrm{aux}}(\widetilde{L}_{\vec{h}}(x, \vec{\xi})) \qquad \vec{\xi} := (\xi_0, \ldots, \xi_{L-1})$$

where

$$\widetilde{L}_{\vec{h}}(x, \vec{\xi}) := \bigotimes_{j=0}^{L-1} \widetilde{L}_{h_j}(x, \xi_j, \xi_{j+1}) = \begin{pmatrix} \widetilde{L}_{\vec{h};1,1}(x, \vec{\xi}) & \widetilde{L}_{\vec{h};1,2}(x, \vec{\xi}) \\ \widetilde{L}_{\vec{h};2,1}(x, \vec{\xi}) & \widetilde{L}_{\vec{h};2,2}(x, \vec{\xi}) \end{pmatrix} \qquad \xi_L := \xi_0.$$

The existence of Baxter vectors $|p_j\rangle$, $p_j := (x, \xi_j, \xi_{j+1}) \in \mathcal{S}_{h_j}$, for all j with the condition $\xi_L = \xi_0$, imposes the constraint of elements $(p_0, \ldots, p_{L-1})$ on the product of surfaces, $\prod_{j=0}^{L-1} \mathcal{S}_{h_j}$, which form a curve $\mathcal{C}_{\vec{h}}$ in $\prod_{j=0}^{L-1} \mathcal{S}_{h_j}$ with the coordinates $(x, \xi_0, \ldots, \xi_{L-1})$ satisfying the relations

$$\mathcal{C}_{\vec{h}} : \xi_j^N = (-1)^N \frac{\xi_{j+1}^N a_j^N - x^N b_j^N}{\xi_{j+1}^N x^N c_j^N - d_j^N} \qquad j = 0, \ldots, L-1. \tag{8}$$

For $p = (p_0, \ldots, p_{L-1}) \in \mathcal{C}_{\vec{h}}$, the Baxter vector $|p\rangle$ is now defined by

$$|p\rangle := |p_0\rangle \otimes \cdots \otimes |p_{L-1}\rangle \in \overset{L}{\otimes}\, \mathbf{C}^N.$$

By the definition of $\widetilde{L}_{\vec{h};j,k}$, the Baxter vector of $\mathcal{C}_{\vec{h}}$ shares the following relations to entries of $\widetilde{L}_{\vec{h}}$ similar to those for $\widetilde{L}_{\vec{h}}(x, \vec{\xi})$ in (4),

$$\widetilde{L}_{\vec{h};1,1}(x, \vec{\xi})|p\rangle = |\tau_- p\rangle \Delta_-(p) \qquad \widetilde{L}_{\vec{h};2,2}(x, \vec{\xi})|p\rangle = |\tau_+ p\rangle \Delta_+(p) \qquad \widetilde{L}_{\vec{h};2,1}(x, \vec{\xi})|p\rangle = 0$$

where $\Delta_\pm$, $\tau_\pm$ are functions and automorphisms of $\mathcal{C}_{\vec{h}}$ defined by

$$\begin{aligned} \Delta_-(x, \xi_0, \ldots, \xi_{L-1}) &= \prod_{j=0}^{L-1} (d_j - x\xi_{j+1} c_j) \\ \Delta_+(x, \xi_0, \ldots, \xi_{L-1}) &= \prod_{j=0}^{L-1} \frac{\xi_j (a_j d_j - x^2 b_j c_j)}{\xi_{j+1} a_j - x b_j} \\ \tau_\pm : (x, \xi_0, \ldots, \xi_{L-1}) &\mapsto (q^{\pm 1} x, q^{-1}\xi_0, \ldots, q^{-1}\xi_{L-1}). \end{aligned} \tag{9}$$

Then the important relation of the transfer matrix on the Baxter vector of the curve $\mathcal{C}_{\vec{h}}$ follows:

$$T_{\vec{h}}(x)|p\rangle = |\tau_- p\rangle\Delta_-(p) + |\tau_+ p\rangle\Delta_+(p) \qquad \text{for} \quad p \in \mathcal{C}_{\vec{h}}. \tag{10}$$

As $T_{\vec{h}}(x)$ are commuting operators for $x \in \mathbf{C}$, a common eigenvector $\langle\varphi|$ is a constant vector of $\overset{L}{\otimes}\mathbf{C}^N$ with an eigenvalue $\Lambda(x) \in \mathbf{C}[x]$. Defining the function $Q(p) = \langle\varphi|p\rangle$ of $\mathcal{C}_{\vec{h}}$ then satisfies the following Bethe equation:

$$\Lambda(x)Q(p) = Q(\tau_-(p))\Delta_-(p) + Q(\tau_+(p))\Delta_+(p) \qquad \text{for} \quad p \in \mathcal{C}_{\vec{h}}. \tag{11}$$

In the rest of this paper we study in detail the above Bethe equation. Before that, we first derive certain functional properties of the eigenvalue $\Lambda(x)$.

Lemma 2. *With the entries $L_{\vec{h};i,j}(x)$ of $L_{\vec{h}}(x)$ in (6), the following properties hold:*

(i) under the interchange of operators, $a_jY \leftrightarrow d_j$, $b_jX \leftrightarrow c_jZ$ for all j, we have the symmetries among $L_{\vec{h};i,j}(x)$s, $L_{\vec{h};1,1}(x) \leftrightarrow L_{\vec{h};2,2}(x)$, $L_{\vec{h};1,2}(x) \leftrightarrow L_{\vec{h};2,1}(x)$;

(ii) the polynomial $L_{\vec{h};i,j}(x)$ is an even or odd function with the parity $(-1)^{i+j}$, and its degree is equal to $2\left[\frac{L+1-\delta_{i,j}}{2}\right] - 1 + \delta_{i,j}$.

Proof. We apply the gauge transformation (7) with $A_j = A_{j+1}$ for all j. When

$$A_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

one obtains the interchange of entries of $L_{h_j}(x)$, hence follows (I). For

$$A_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the corresponding $\widetilde{L}_{h_j}(x)$ is equal to $L_{h_j}(-x)$, which implies the parity of $L_{\vec{h};i,j}(x)$. The determination of the degree of $L_{\vec{h};i,j}(x)$ can be obtained by certain suitable choices of the values of h_j. □

From the definition of $T_{\vec{h}}(x)$, $\Lambda(x)$, one can easily obtain the following result.

Proposition 1. *The transform matrix $T_{\vec{h}}(x)$ is an operator-coefficient even polynomial of x with degree $2\left[\frac{L}{2}\right]$, which is invariant under the substitutions in lemma 2 (i). The constant term of $T_{\vec{h}}(x)$ is given by*

$$T_0 := T_{\vec{h}}(0) = \prod_{j=0}^{L-1} a_j \bigotimes^{L} Y + \prod_{j=0}^{L-1} d_j. \tag{12}$$

Subsequently, the polynomial $\Lambda(x)$ in (11) is an even function of degree $\leqslant 2\left[\frac{L}{2}\right]$ with $\Lambda(0) = q^l \prod_{j=0}^{L-1} a_j + \prod_{j=0}^{L-1} d_j$ for some l.

From the above proposition, we have $T_{\vec{h}}(x) = \sum_{j=0}^{[\frac{L}{2}]} T_{2j}x^{2j}$ with T_0 given by (12). With a further study of the expressions of T_{2j}, one can show that they form a commuting family of operators, whose proof we will not present here. Instead, as an illustration of this fact and also for further use in this paper, we list below the explicit form of T_2 for $L = 2, 3$ in $T_{\vec{h}}(x) = T_0 + x^2T_2$, where the commutation relation of T_2 and T_0 is easily verified:

$$\begin{aligned}
L = 2 \qquad & T_2 = b_0c_1X \otimes Z + c_0b_1Z \otimes X \\
L = 3 \qquad & T_2 = b_0c_1a_2X \otimes Z \otimes Y + a_0b_1c_2Y \otimes X \otimes Z + c_0a_1b_2Z \otimes Y \otimes X \\
& \qquad + c_0b_1d_2Z \otimes X \otimes I + d_0c_1b_2I \otimes Z \otimes X + b_0d_1c_2X \otimes I \otimes Z.
\end{aligned} \tag{13}$$

The above T_2 for $L = 3$ can be written as the Hofstadter-type Hamiltonian H_{FK} of Faddeev and Kashaev described in section 1 (for the exact identification, see [12])[6]. In equation (11), $Q(p)$ is a rational function of $\mathcal{C}_{\vec{h}}$ with poles. As the functions of $\mathcal{C}_{\vec{h}}$, $\xi_{j+1}^N x^N c_j^N - d_j^N$ and $\left(\xi_{j+1}^N a_j^N - x^N b_j^N\right)(-\xi_j)^{-N}$ are the same, by the description of the Baxter vector, the poles of $Q(p)$ are contained in the divisor of $\mathcal{C}_{\vec{h}}$ defined by

$$\prod_{j=0}^{L-1}\left(\xi_j^N x^N c_j^N - d_j^N\right) = 0.$$

Hence the understanding of the Bethe solutions of (11) relies heavily on the function theory of $\mathcal{C}_{\vec{h}}$; the algebraic geometry of the curve will play a key role in the complexity of the problem. We shall specify the spectral curves in further discussion. For our purpose, in the rest of this paper we shall only consider the situation when N is an odd integer, and denote the integer $\left[\frac{N}{2}\right]$ by M:

$$N = 2M + 1.$$

4. The rational degenerated Bethe equation

For the next four sections, the spectral curve $\mathcal{C}_{\vec{h}}$ will always be the rational curve under the following assumption of degeneration:

$$a_j = q^{-1} d_j \qquad b_j = q^{-1} c_j \qquad \text{for} \quad j = 0, \ldots, L-1. \tag{14}$$

To make our presentation self-contained, we shall briefly review in this section the general procedure of reducing the Bethe equation on $\mathcal{C}_{\vec{h}}$ to a polynomial equation, that originally appeared in [12], and present an explicit form of the Baxter vector which will be convenient for further use. By replacing c_j, d_j by $\frac{c_j}{d_j}, 1$, we may assume $d_j = 1$ for all j. For the convenience of mathematical discussion, also suitable for physical applications, we shall assume that the parameters c_j are all generic. The solutions for ξ_j in (8) are given by

$$\xi_0^N = \cdots = \xi_{L-1}^N = \pm 1$$

which possess the structure of a finite Abelian group. Therefore $\mathcal{C}_{\vec{h}}$ is the union of disjoint copies of the x-(complex) line indexed by this finite group. Instead of working on the curve $\mathcal{C}_{\vec{h}}$, the following $\tau_\pm$-invariant subset of $\mathcal{C}_{\vec{h}}$ will be sufficient for our discussion of the Bethe equation:

$$\mathcal{C} := \{(x, \xi_0, \ldots, \xi_{L-1}) \mid \xi_0 = \cdots = \xi_{L-1} = q^l, l \in \mathbf{Z}_N\}.$$

The curve $\mathcal{C}$ will be identified with $\mathbf{P}^1 \times \mathbf{Z}_N$:

$$\mathcal{C} = \mathbf{P}^1 \times \mathbf{Z}_N \qquad (x, q^l, \ldots, q^l) \longleftrightarrow (x, l).$$

The Baxter vectors are now labelled by $|x, l\rangle = \otimes_{j=0}^3 |x, l\rangle_j$, where $|x, l\rangle_j \in \mathbf{C}^N$ is defined by the relations

$$\langle 0|x, l\rangle_j = 1 \qquad \frac{\langle k|x, l\rangle_j}{\langle k-1|x, l\rangle_j} = \frac{q^{2k-1}\left(1 - x c_j q^{-l-2k}\right)}{\left(1 - x c_j q^{l+2k}\right)} \qquad k \in \mathbf{Z}_N.$$

We shall use the bold letter $\mathbf{k}$ to denote a multi-index vector $\mathbf{k} = (k_0, \ldots, k_{L-1})$ with $k_j \in \mathbf{Z}$; the square-length of $\mathbf{k}$ is defined by $|\mathbf{k}|^2 := \sum_{j=0}^{L-1} k_j^2$. The component-expression of the vector $|x, l\rangle$ is given by

$$\langle \mathbf{k}|x, l\rangle = q^{|\mathbf{k}|^2} \prod_{j=0}^{L-1} \prod_{i=1}^{k_j} \frac{1 - x c_j q^{-l-2\mathrm{i}}}{1 - x c_j q^{l+2\mathrm{i}}} \qquad k_j > 0. \tag{15}$$

[6] The operators X, Y, Z, S, T, U in [12] correspond to Z, X, Y, U, V, W respectively in this paper.

Equation (10) takes the form

$$T(x)|x,l\rangle = |q^{-1}x, l-1\rangle\Delta_-(x,l) + |qx, l-1\rangle\Delta_+(x,l) \tag{16}$$

where $\Delta_\pm$ are given by

$$\Delta_-(x,l) = \prod_{j=0}^{L-1}\left(1 - xc_jq^l\right) \qquad \Delta_+(x,l) = \prod_{j=0}^{L-1}\frac{1-x^2c_j^2}{1-xc_jq^{-l}}.$$

As in [12], we introduce the following two functions on the curve $\mathcal{C}$:

$$f^e(x,2n) = \prod_{j=0}^{L-1}\prod_{k=0}^{n}\frac{1-xc_jq^{-2(n-k)}}{1-xc_jq^{2(n-k)}} \qquad f^o(x,2n+1) = \prod_{j=0}^{L-1}\prod_{k=0}^{n}\frac{1-xc_jq^{-1-2(n-k)}}{1-xc_jq^{1+2(n-k)}} \tag{17}$$

by which we define the vectors

$$|x\rangle_m^e = \sum_{n=0}^{N-1}|x,2n\rangle f^e(x,2n)\omega^{mn} \qquad |x\rangle_m^o = \sum_{n=0}^{N-1}|x,2n+1\rangle f^o(x,2n+1)\omega^{mn}. \tag{18}$$

By computation, the following relations hold for $n \in \mathbf{Z}_N$:

$$\frac{f^e(x,2n)}{f^o(q^{\pm1}x,2n-1)}\Delta_\pm(x,2n) = \Delta_\pm(x,0)$$
$$\frac{f^o(x,2n-1)}{f^e(xq^{\pm1},2n-2)}\Delta_\pm(x,2n-1) = \Delta_\pm(x,-1).$$

Then (16) becomes the system of equations,

$$T(x)|x\rangle\rangle_m = |q^{-1}x\rangle\rangle_m D_m^-(x) + |qx\rangle\rangle_m D_m^+(x) \qquad m \in \mathbf{Z}_N$$

where

$$|x\rangle\rangle_m = \left(|x\rangle_m^e, |x\rangle_m^o\right) \qquad D_m^\pm(x) = \begin{pmatrix} 0 & \Delta_\pm(x,-1) \\ \omega^m\Delta_\pm(x,0) & 0\end{pmatrix}.$$

In what follows, it is convenient to use the notation of a shifted factorial:

$$(a;\alpha)_0 = 1 \qquad (a;\alpha)_n = (1-a)(1-a\alpha)\cdots(1-a\alpha^{n-1}) \qquad n \in \mathbf{Z}_{>0}.$$

From (15) and (17), we have

$$f^e(x,2n)\langle\mathbf{k}|x,2n\rangle = q^{|\mathbf{k}|^2}\prod_{j=0}^{L-1}\frac{\left(xc_j;\omega^{-1}\right)_{k_j+n+1}}{(xc_j;\omega)_{k_j+n+1}}$$
$$f^o(x,2n+1)\langle\mathbf{k}|x,2n+1\rangle = q^{|\mathbf{k}|^2}\prod_{j=0}^{L-1}\frac{\left(xc_jq^{-1};\omega^{-1}\right)_{k_j+n+1}}{(xc_jq;\omega)_{k_j+n+1}}.$$

Note that each ratio term on the right-hand side of the above is defined when the lower index $k_j + n + 1$ is positive; however, its value depends only on the class modular N. So we shall use the same notation for an arbitrary integer index n by defining the value equal to that of any positive representative of the class involved in $\mathbf{Z}_N$, and will keep this convention in what follows. From (18), we have

$$\langle\mathbf{k}|x\rangle_m^e = q^{|\mathbf{k}|^2}\sum_{n\in\mathbf{Z}_N}\omega^{mn}\prod_{j=0}^{L-1}\frac{\left(xc_j;\omega^{-1}\right)_{k_j+n+1}}{(xc_j;\omega)_{k_j+n+1}}$$
$$\langle\mathbf{k}|x\rangle_m^o = q^{|\mathbf{k}|^2}\sum_{n\in\mathbf{Z}_N}\omega^{mn}\prod_{j=0}^{L-1}\frac{\left(xc_jq^{-1};\omega^{-1}\right)_{k_j+n+1}}{(xc_jq;\omega)_{k_j+n+1}}. \tag{19}$$

We proceed with the diagonalizing procedure on the matrix $D_m^{\pm}(x)$ by a gauge transformation using an invertible 2×2 matrix $U_m(x)$:

$$|x\rangle\rangle_m \mapsto |x\rangle_m := |x\rangle\rangle_m U_m(x) \qquad D_m^{\pm}(x) \mapsto U_m(q^{\pm 1}x)^{-1} D_m^{\pm}(x) U_m(x).$$

We shall choose the matrix $U_m(x)$ of the form[7]

$$U_m(x) = \begin{pmatrix} q^{-m}u(qx) & u(qx) \\ u(x) & -q^m u(x) \end{pmatrix}.$$

Then the diagonalizable criterion of $D_m^{\pm}(x)$ for all m is equivalent to the following equation of $u(x)$:

$$\frac{u(\omega x)}{u(x)} = \prod_{j=0}^{L-1} \frac{1 - c_j x}{1 - c_j x q}. \tag{20}$$

Note that the right-hand side of the above form is equal to $\frac{\Delta_+(x,-1)}{\Delta_+(x,0)}$, which is the same as $\frac{\Delta_-(qx,-1)}{\Delta_-(qx,0)}$. The resulting expression of $D_m^{\pm}(x)$ becomes

$$D_m^-(x) = q^m \Delta_-(x,-1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad D_m^+(x) = q^m \Delta_+(x,0) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With the notation

$$|x\rangle_m = (|x\rangle_m^+, |x\rangle_m^-) \qquad (Q_m^+(x), Q_m^-(x)) = \langle \varphi | x \rangle_m$$

one has

$$|x\rangle_m^+ = |x\rangle_m^e q^{-m} u(qx) + |x\rangle_m^o u(x) \qquad |x\rangle_m^- = |x\rangle_m^e u(qx) - |x\rangle_m^o q^m u(x).$$

The Bethe equation (11) now takes the form

$$\pm q^{-m} \Lambda(x) Q_m^{\pm}(x) = \prod_{j=0}^{L-1} \left(1 - x c_j q^{-1}\right) Q_m^{\pm}(xq^{-1}) + \prod_{j=0}^{L-1} (1 + x c_j) Q_m^{\pm}(xq). \tag{21}$$

5. Solutions of the rational Bethe equation

In this section, we advance the discussion of the last section to obtain the explicit solutions of the Bethe equation (21) for $L \leqslant 3$, from which a special case will be shown in the next section to coincide with that in [12, 23].

Lemma 3. *The general solutions of the rational function $u(x)$ for (20) are given by*

$$u(x) = R(x^N) \prod_{j=0}^{L-1} (c_j x; \omega)_{M+1}^{-1}$$

where $R(x^N)$ is a rational function of x^N.

Proof. Note that the ratio of any two solutions of (20) is a rational function $r(x)$ with the relation $r(\omega x) = r(x)$, which is equivalent to $r(x) = R(x^N)$ for a rational function $R(x^N)$ of x^N. Therefore it suffices to show that $\prod_{j=0}^{L-1} (c_j x; \omega)_{M+1}^{-1}$ is a solution of (20), which is easily seen by $q = \omega^{M+1}$. □

[7] The form of the gauge transformation matrix here is slightly different from that of (5.15) in [12] by rearrangement of the entries. As we are not able to produce the required formulation through the expression in [12], we wonder if there might be a misprint in it.

By the expressions of $\langle \mathbf{k}|x\rangle_m^e$, $\langle \mathbf{k}|x\rangle_m^o$ in (19), in order to have the polynomial functions of $Q_m^{\pm}(x)$ in (21), we choose the following gauge function $u(x)$ by setting $R(x^N) = \prod_{j=0}^{L-1}\left(1-x^N c_j^N\right)^2$ in lemma 3:

$$u(x) = \prod_{j=0}^{L-1}\left(1-x^N c_j^N\right)\left(xc_jq;q^2\right)_M. \tag{22}$$

The polynomial $Q_m^{\pm}(x)$ has the degree at most equal to that of $u(x)$, which is $(3M+1)L$. By proposition 1, one requires the polynomial solutions $Q_m^{\pm}(x)$, $\Lambda(x)$ of the Bethe equation (21) with the constraints,

$$\deg Q_m^{\pm}(x) \leqslant (3M+1)L$$
$$\deg \Lambda(x) \leqslant 2\left[\frac{L}{2}\right] \qquad \Lambda(x) = \Lambda(-x) \quad \Lambda(0) = q^l + 1 \text{ for some } l.$$

Remark. For another choice of gauge function $u(x)$, only the function $Q_m^{\pm}(x)$ differs by a multiple of a certain rational function of x^N, which has no effect as far as the Bethe equation is concerned.

Lemma 4. *Let q be a primitive Nth root of unity, k, l be integers with $q^k + q^l \in \mathbf{R}$. Then $q^{k+l} = 1$.*

Proof. By the odd property of N, 1 is the only real number among the Nth roots of unity. We may assume that $1, q^k, q^l$ are three distinct numbers. The following conditions are equivalent:

$$q^k + q^l \in \mathbf{R} \iff q^k - q^{-l} \in \mathbf{R} \iff q^{k+l} - 1 \in q^l\mathbf{R}.$$

By interchanging k and l in the above relations, one concludes that $(q^{k+l}-1) \in q^k\mathbf{R} \cap q^l\mathbf{R}$. By $q^k \neq \pm q^l$, $q^k\mathbf{R} \cap q^l\mathbf{R}$ consists of only the zero element. Hence $q^{k+l} = 1$. □

For the Bethe equation (21), one needs only to consider the plus part of the equation because of the following result.

Proposition 2. *For $m \in \mathbf{Z}_N$, we have $Q_m^-(x) = 0$, $|x\rangle_m^- = \vec{0}$, and*

$$|x\rangle_m^+ = 2q^{-m}|x\rangle_m^e u(qx) = 2|x\rangle_m^o u(x). \tag{23}$$

Proof. Let $Q_m^-(x)$ be a non-zero polynomial solution of (21) for some $\Lambda(x)$ with $\Lambda(0) = q^l+1$, and write $Q_m^-(x) = x^r Q_m^{-*}(x)$ with $Q_m^{-*}(0) \neq 0$. By comparing the x^r-coefficients of (21), we have $-q^{-m}\Lambda(0)Q_m^{-*}(0) = (q^{-r}+q^r)Q_m^{-*}(0)$, hence

$$-q^{-m}(1+q^l) = q^{-r} + q^r.$$

By lemma 4, $q^l = q^{2m}$, which implies that $q^r = -q^{\pm m}$, a contradiction to the odd property of the integer N. Therefore the only solution $Q_m^-(x)$ for the negative part of (21) is the trivial one. Since $Q_m^-(x)$ is of the form $\langle\varphi|x\rangle_m^-$ for an eigenvector $\langle\varphi|$ of $T(x)$ in $\overset{L}{\otimes}\,\mathbf{C}^{N*}$, and all such vectors $\langle\varphi|$ form a basis of $\overset{L}{\otimes}\,\mathbf{C}^{N*}$, hence $|x\rangle_m^- = \vec{0}$ for all m. Then follows relation (23). □

We now derive some general properties of the solutions $Q_m^+(x)$ of (21).

Lemma 5. *For a polynomial $\Lambda(x)$ with $\Lambda(0) = q^l + 1$, the necessary and sufficient condition of $\Lambda(x)$ for the existence of a non-zero polynomial solution $Q_m^+(x)$ of equation (21) with the eigenvalue $\Lambda(x)$ is given by the relation $q^l = q^{2m}$. In this situation, with $Q_m^+(x) = x^r Q_m^{+*}(x)$ and $Q_m^{+*}(0) \neq 0$, one has $q^r = q^{\pm m}$.*

Proof. Let $Q_m^+(x)$ be a non-zero solution of (21) and write $Q_m^+(x) = x^r Q_m^{+*}(x)$ with $Q_m^{+*}(0) \neq 0$. By the same argument as in proposition 2, we have

$$q^{-m}(1+q^l) = q^{-r} + q^r$$

hence $q^l = q^{2m}$ and $q^r = q^{\pm m}$ by lemma 4. The 'sufficient' part of the statement remains to be shown. For $q^l = q^{2m}$, by (19) and (23), one concludes that $|x\rangle_m^+$ cannot be identically zero. We claim that the solution $Q_m^+(x)$ in (21) has non-trivial solutions. Otherwise, this implies that $|x\rangle_m^+$ is always the zero vector for all x by the same argument as in proposition 2, hence a contradiction. □

Proposition 3. *Let m be an integer between 0 and M, and $Q_m^+(x)$, $Q_{N-m}^+(x)$ solutions of (21) for m, $N-m$ respectively which arise from the evaluation of eigenvectors $\langle\varphi|$ of $T_{\bar h}^-(x)$ on the Baxter vector. Then $Q_m^+(x)$, $Q_{N-m}^+(x)$ are elements in $x^m \prod_{j=0}^{L-1}\left(1-x^N c_j^N\right)\mathbf{C}[x]$.*

Proof. The divisibility of $Q_m^+(x)$, $Q_{N-m}^+(x)$ by x^m follows easily from lemma 5, so only the factor $\prod_{j=0}^{L-1}\left(1-x^N c_j^N\right)$ remains to be verified. As $Q_l^+(x)$ is of the form $\langle\varphi|x\rangle_l^+$ for some vector $\langle\varphi|$ in $\overset{L}{\otimes}\,\mathbf{C}^{N*}$, and by (19) and (23), it suffices to show the following divisibility of polynomials:

$$\prod_{j=0}^{L-1}(xc_j;w)_{M+1} \mid u(qx)\prod_{j=0}^{L-1}\frac{\left(xc_j;\omega^{-1}\right)_{k_j+n+1}}{(xc_j;\omega)_{k_j+n+1}}$$
$$\prod_{j=0}^{L-1}(xc_j\omega^{M+1};w)_{M} \mid u(x)\prod_{j=0}^{L-1}\frac{\left(xc_jq^{-1};\omega^{-1}\right)_{k_j+n+1}}{(xc_jq;\omega)_{k_j+n+1}}.$$

By the form of $u(x)$ in (22) and the relation $q\omega^{M+1+j} = \omega^{j+1}$ for $j \in \mathbf{Z}$, the above relations are easily seen. □

For our purpose, the functions $Q_m^+(x)$ that we shall consider are only those arising from eigenvectors of the transfer matrix, hence $Q_m^+(x)$ is in the form of proposition 3. For the rest of this paper, the letter m will always denote an integer between 0 and M,

$$0 \leqslant m \leqslant M.$$

We shall conduct our discussion of the plus part of equation (21) for the sectors m, $N-m$ simultaneously by introducing the polynomials $\Lambda_m(x)$, $Q_m(x)$ via the relation

$$\left(\Lambda_m(x), x^m\prod_{j=0}^{L-1}\left(1-x^Nc_j^N\right)Q_m(x)\right) = \left(q^{-m}\Lambda(x), Q_m^+(x)\right), \left(q^m\Lambda(x), Q_{N-m}^+(x)\right). \tag{24}$$

Then relations (21) for both the m and $N-m$ sectors are reduced to the following:

$$\Lambda_m(x)Q_m(x) = q^{-m}\prod_{j=0}^{L-1}(1-xc_jq^{-1})Q_m(xq^{-1}) + q^m\prod_{j=0}^{L-1}(1+xc_j)Q_m(xq) \tag{25}$$

where $Q_m(x)$, $\Lambda_m(x)$ are polynomials with

$$\deg Q_m(x) \leqslant ML - m$$
$$\deg \Lambda_m(x) \leqslant 2\left[\frac{L}{2}\right] \qquad \Lambda_m(x) = \Lambda_m(-x) \quad \Lambda(0) = q^m + q^{-m}.$$

The general mathematical problem will be the structure of the solution space of the Bethe equation (25) for a given positive integer L. First, we derive a detailed answer of the Bethe solutions for the simplest case $L = 1$.

$L = 1$. We have $\Lambda_m(x) = q^m + q^{-m}$, and $\deg Q_m(x) \leqslant M - m$.

Theorem 1. *The solutions $Q_m(x)$ of $(25)_{L=1}$ form a one-dimensional vector space generated by the following polynomial of degree $M - m$:*

$$B_m(x) := 1 + \sum_{j\geqslant 1} \left(\prod_{i=1}^{j} \frac{q^{m+i-1} - q^{-m-i}}{q^m + q^{-m} - q^{-m-i} - q^{m+i}} \right) (xc_0)^j. \tag{26}$$

(Note that the coefficients in the above expression are zero for $j > M - m$.)

Proof. Write

$$Q_m(x) = \sum_{j=0}^{M-m} \beta_j (xc_0)^j.$$

Then $(25)_{L=1}$ is equivalent to the following system of equations of β_j:

$$(q^m + q^{-m} - q^{-m-j} - q^{m+j})\beta_j = (q^{m+j-1} - q^{-m-j})\beta_{j-1} \qquad j \in \mathbf{Z}_{\geqslant 0} \tag{27}$$

where β_k is defined to be zero for k not between 0 and $M - m$. As the values of $q^m + q^{-m} - q^{-m-j} - q^{m+j}, q^{-m-j} - q^{m+j-1}$ are all non-zero, the polynomial $Q_m(x)$ is determined by β_0 (or equivalently β_{M-m}) through the recursive relations (27). With $\beta_0 = 1$, this provides the basis element $B_m(x)$. □

Corollary 1. *The vector space of all polynomial solutions of $(25)_{L=1}$ (without the restriction of the degree of $Q_m(x)$) is $\mathbf{C}[x^N]B_m(x)$.*

Proof. By using the fact

$$q^{m+j-1} - q^{-m-j} = 0 \iff j \equiv M - m + 1 \pmod N$$
$$q^m + q^{-m} - q^{m+j} - q^{-m-j} = 0 \iff j \equiv 0,\ N - 2m \pmod N$$

and relation (27) for those j with $M - m + 1 + lN < j \leqslant (l + 1)N$ ($l \in \mathbf{Z}_{\geqslant 0}$), any solution $\sum_{k\geqslant 0} \beta_j (c_0x)^k$ of $(25)_{L=1}$ must have $\beta_k = 0$ except $k \equiv 0, \ldots, M - m \pmod N$, and hence is an element of $\mathbf{C}[x^N]B_m(x)$ by theorem 1. □

For the Bethe equation (25) with $L > 1$, by the scaling of the variables,

$$x \mapsto \lambda^{-1}x \qquad c_j \mapsto \lambda c_j \qquad \text{for} \quad \lambda \in \mathbf{C}^*$$

we may assume that the x^j-coefficients of polynomials, $\Lambda_m(x), Q_m(x)$, are always homogenous functions of $c_0, \ldots, c_{L-1}$ with the degree j. As (25) is invariant under permutations of c_j, the coefficients of the polynomials of x involved in (25) depend only on the elementary symmetric functions of c_j,

$$s_j = \sum_{i_1<\cdots<i_j} c_{i_1} \cdots c_{i_j} \qquad \text{for} \quad j = 1, \ldots, L.$$

We shall denote

$$Q_m(x) = \sum_{j=0}^{d} \alpha_j x^j \qquad d := \deg Q_m(x) \quad (\leqslant LM - m)$$

and define α_j to be zero for j not between 0 and d. For the rest of this section, we shall only consider the case $L = 2, 3$.

$L = 2$. We have only two elementary symmetric functions of c_j:

$$s_1 = c_0 + c_1 \qquad s_2 = c_0c_1.$$

Lemma 6. *Let n be an odd positive integer, A an $n \times n$ matrix with the complex entries $a_{i,j}$ satisfying the relations*

$$a_{i,j} = (-1)^{i+j+1} a_{n-j+1,n-i+1} \qquad \text{for} \quad 1 \leqslant i, j \leqslant n.$$

Then A is a degenerated matrix.

Proof. Write $n = 2h + 1$. The determinant of A can be expressed by

$$\begin{aligned}\det(A) &= \sum_\sigma \mathrm{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)} = \sum_\sigma \mathrm{sgn}(\sigma) \prod_{i=1}^n (-1)^{i+\sigma(i)+1} a_{n-\sigma(i)+1,n-i+1} \\ &= \mathrm{sgn}(\sigma_0)(-1)^{h+1} \sum_{\sigma'} \mathrm{sgn}(\sigma') \prod_{j=1}^n a_{\sigma'(j),j} = \mathrm{sgn}(\sigma_0)(-1)^{h+1} \det(A)\end{aligned}$$

where the indices σ, σ' run through all permutations of $\{1, \ldots, n\}$, and σ_0 is the one defined by $\sigma_0(i) = n - i + 1$. By $\mathrm{sgn}(\sigma_0) = (-1)^h$, we have $\det(A) = 0$. □

In equation $(25)_{L=2}$, $\Lambda_m(x)$ is an even polynomial of degree $\leqslant 2$, and $\deg Q_m(x) \leqslant 2M - m$. By comparing the coefficients of the highest degree of x in (25), we have

$$\Lambda_m(x) = (q^{m+d} + q^{-m-d-2}) x^2 s_2 + q^m + q^{-m}.$$

For $k \in \mathbf{Z}$, we define

$$\begin{aligned} v_k &= q^k + q^{-k} - q^m - q^{-m} \\ \delta_k &= (q^{k-1} - q^{-k}) s_1 \\ u_k &= (q^{k-2} + q^{-k} - q^{m+d} - q^{-m-d-2}) s_2. \end{aligned} \tag{28}$$

Then $(25)_{L=2}$ is equivalent to the system of linear equations of α_j,

$$v_{m+j}\alpha_j + \delta_{m+j}\alpha_{j-1} + u_{m+j}\alpha_{j-2} = 0 \qquad j \in \mathbf{Z}_{\geqslant 0}. \tag{29}$$

In fact, the non-trivial relations in the above system are those for j between 1 and $d+1$, hence the matrix form of (29) is given by

$$\begin{pmatrix} \delta_{m+d+1} & u_{m+d+1} & 0 & \cdots & 0 & 0 \\ v_{m+d} & \delta_{m+d} & u_{m+d} & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & v_{m+2} & \delta_{m+2} & u_{m+2} \\ 0 & \cdots & & 0 & v_{m+1} & \delta_{m+1} \end{pmatrix} \begin{pmatrix} \alpha_d \\ \alpha_{d-1} \\ \vdots \\ \vdots \\ \vdots \\ \alpha_0 \end{pmatrix} = \vec{0}. \tag{30}$$

Theorem 2. *Equation $(25)_{L=2}$ has a non-trivial solution $Q_m(x)$ if and only if* $\deg Q_m(x) = M - m + m'$ *for $0 \leqslant m' \leqslant M$. For each such m', the eigenvalue $\Lambda_m(x)$ in $(25)_{L=2}$ is equal to $\Lambda_{m,m'}(x) := q^{\frac{1}{2}}(q^{m'-1} + q^{-m'-2})x^2 s_2 + q^m + q^{-m}$, and the corresponding solutions of $Q_m(x)$ form a one-dimensional space generated by a polynomial $B_{m,m'}(x)$ of degree $M - m + m'$ with $B_{m,m'}(0) = 1$. (Here $q^{\frac{1}{2}} := q^{M+1}$.)*

Proof. Denote $d = \deg Q_m(x)$. Among those v_j of the entries of the square matrix (30), there is at most one zero term which is given by $v_{N-2m} = 0$. If $Q_m(0) = 0$, this implies that $Q_m(x) = x^{N-2m} Q_m^*(x)$ with $Q_m^*(0) \neq 0$, and each coefficient of the polynomial $Q_m^*(x)$ is expressed by a $Q_m^*(0)$-multiple of a certain polynomial of c_0, c_1. By setting $c_1 = 0$, $x^{N-2m} Q_m^*(x) Q_m^*(0)^{-1}$ gives rise to a solution of $(25)_{L=1}$, which contradicts the conclusion of corollary 1. Therefore $Q_m(0) \neq 0$. If d is less than $M - m$, again by setting $c_1 = 0$, the function $Q_m(x)$ gives rise to a solution of $(25)_{L=1}$ of degree $<M - m$, a contradiction to theorem 1. Hence $d = M - m + m'$ for $0 \leqslant m' \leqslant M$. It remains to be shown for each such m' that the solutions $Q_m(x)$ form a one-dimensional vector space. As any non-trivial solution $Q_m(x)$ must have $Q_m(0) \neq 0$, this implies the injectivity of the following linear functional of the solution space:

$$Q_m(x) \mapsto Q_m(0) \in \mathbf{C}.$$

So one needs only to show the existence of a non-trivial solution $Q_m(x)$, which is equivalent to the degeneracy of the square matrix of size $d + 1$ on the left-hand side of (30). Write this square matrix in the form

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \tag{31}$$

where C is a $(d - 2m') \times (2m' + 1)$ matrix, A, B are the tri-diagonal square matrices of size $2m' + 1$, $d - 2m'$ respectively. The explicit form of A is given by

$$A = \begin{pmatrix} \delta_{M+1+m'} & u_{M+1+m'} & 0 & \cdots & 0 & 0 \\ v_{M+m'} & \delta_{M+m'} & u_{M+m'} & 0 & \ddots & 0 \\ 0 & v_{M+m'-1} & \delta_{M+m'-1} & u_{M+m'-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & v_{M+2-m'} & \delta_{M+2-m'} & u_{M+2-m'} \\ 0 & \cdots & & 0 & v_{M+1-m'} & \delta_{M+1-m} \end{pmatrix}.$$

From (28), we have $v_j = v_{N-j}, \delta_j = -\delta_{N+1-j}, u_j = u_{N+2-j}$. This implies that matrix A satisfies the condition of lemma 6, hence $\det(A) = 0$. Therefore the square matrix (31) has a zero determinant. □

Remark. From the above theorem, the following data are in one-to-one correspondence with integers m, m' between 0 and M:

$$(m, m') \longleftrightarrow \Lambda_{m,m'}(x) \longleftrightarrow B_{m,m'}(x).$$

The characterization of $B_{m,m'}(x)$ is given as a (unique) polynomial with $\deg B_{m,m'}(x) = M - m + m'$ and $B_{m,m'}(0) = 1$, whose coefficients α_j satisfy equation (29) with

$$\begin{aligned} v_k &= q^k + q^{-k} - q^m - q^{-m} \\ \delta_k &= (q^{k-1} - q^{-k})s_1 \\ u_k &= \left(q^{k-2} + q^{-k} - q^{m'-\frac{1}{2}} - q^{-m'-\frac{5}{2}}\right) s_2. \end{aligned} \tag{32}$$

L = 3. There are three elementary symmetric functions of c_j,

$$s_1 = c_0 + c_1 + c_2 \qquad s_2 = c_0c_1 + c_1c_2 + c_2c_0 \qquad s_3 = c_0c_1c_3.$$

A non-trivial solution $Q_m(x)$ of $(25)_{L=3}$ has the degree $d \leqslant 3M - m$, and $\Lambda_m(x)$ is of the form

$$\Lambda_m(x) = \lambda_m x^2 + q^m + q^{-m}. \tag{33}$$

Note that λ_m is a homogeneous function of c_j of degree 2. For $k \in \mathbf{Z}$, we define

$$\begin{aligned} w_k &= q^k + q^{-k} - q^m - q^{-m} \\ v_k &= (q^{k-1} - q^{-k})s_1 \\ \delta_k &= (q^{k-2} + q^{-k})s_2 \\ u_k &= (q^{k-3} - q^{-k})s_3. \end{aligned}$$

Equation $(25)_{L=3}$ is equivalent to the system of linear equations of α_j,

$$w_{m+j}\alpha_j + v_{m+j}\alpha_{j-1} + (\delta_{m+j} - \lambda_m)\alpha_{j-2} + u_{m+j}\alpha_{j-3} = 0 \qquad j \in \mathbf{Z}_{\geqslant 0}. \tag{34}$$

The non-trivial relations of the above system are those for j between 1 and $d+3$.

Lemma 7. *Let $Q_m(x)$ be a non-trivial polynomial solution of $(25)_{L=3}$ for some $\Lambda_m(x)$. Then the degree of $Q_m(x)$ is equal to $3M - m$ with $Q_m(0) \neq 0$.*

Proof. First, we note that for j between $m+1$ and $3M$, the only possible w_j with zero value are given by

$$w_{N-m} = w_{m+N} = 0.$$

Let r be the zero multiplicity of $Q_m(x)$ at $x = 0$. If $Q_m(0) = 0$, by (34) we have $r = N - 2m$ or N. The polynomial $\alpha_r^{-1} Q_m(x)$ with $c_1 = c_2 = 0$ is a non-trivial solution of $(25)_{L=1}$ with zero multiplicity r. By corollary 1, r must be equal to N and the degree of $Q_m(x)$ is at least $N + M - m$, which contradicts our assumption, $d \leqslant 3M - m$. Therefore $Q_m(0) \neq 0$. By the relation of $j = d+3$ in (34), we have $q^{m+d} = q^{-m-d-3}$, hence the only possible values of d are $M - m - 1$, $3M - m$. If $d = M - m - 1$, by $w_j \neq 0$ for $m + 1 \leqslant j < M$, the function $Q_m(x)$ with $c_1 = c_2 = 0$ gives rise to a non-trivial solution of $(25)_{L=1}$ with degree $<M - m$, a contradiction to theorem 1. Therefore $d = 3M - m$. □

By the above lemma, the 'eigenfunction' $Q_m(x)$ for an eigenvalue $\Lambda_m(x)$ is unique up to a non-zero constant, hence there is a one-to-one correspondence between the eigenvalues and eigenstates of the Bethe equation $(34)_{L=3}$ for a given m. By $d = 3M - m$, the $(d+3)$th relation in system (34) is redundant, hence the matrix form of (34) becomes

$$\begin{pmatrix} A - \lambda_m & 0 \\ C & B \end{pmatrix} \begin{pmatrix} \alpha_d \\ \vdots \\ \vdots \\ \alpha_0 \end{pmatrix} = \vec{0} \qquad d = 3M - m \tag{35}$$

where A is the $N \times N$ matrix,

$$A = \begin{pmatrix} \delta'_{N-1} & u'_{N-1} & 0 & \cdots & & 0 & 0 \\ v'_{N-2} & \delta'_{N-2} & u'_{N-2} & 0 & \ddots & & \vdots \\ w'_{N-3} & v'_{N-3} & \delta'_{N-3} & u'_{N-3} & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & w'_1 & v'_1 & \delta'_1 & u'_1 \\ 0 & \cdots & & 0 & w'_0 & v'_0 & \delta'_0 \end{pmatrix} \tag{36}$$

with the entries defined by

$$w'_k = q^{k+\frac{3}{2}} + q^{-k-\frac{3}{2}} - q^m - q^{-m} \qquad v'_k = \left(q^{k+\frac{1}{2}} - q^{-k-\frac{3}{2}}\right) s_1$$
$$\delta'_k = \left(q^{k-\frac{1}{2}} + q^{-k-\frac{3}{2}}\right) s_2 \qquad u'_k = \left(q^{k-\frac{3}{2}} - q^{-k-\frac{3}{2}}\right) s_3$$

and B, C are the following matrices:

$$B = \begin{pmatrix} \delta_{M+1} - \lambda_m & u_{M+1} & 0 & \cdots & 0 & 0 \\ v_M & \delta_M - \lambda_m & u_M & 0 & \ddots & \vdots \\ w_{M-1} & v_{M-1} & \delta_{M-1} - \lambda_m & u_{M-1} & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & w_{m+3} & v_{m+3} & \delta_{m+3} - \lambda_m & u_{m+3} \\ 0 & \cdots & 0 & w_{m+2} & v_{m+2} & \delta_{m+2} - \lambda_m \\ 0 & \cdots & 0 & 0 & w_{m+1} & v_{m+1} \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & \cdots & \cdots & 0 & w_{M+1} & 0 \\ 0 & \cdots & & & 0 & w_M \\ \vdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & & \cdots & \cdots & 0 \end{pmatrix} \qquad w_M = w_{M+1} = q^{\frac{1}{2}} + q^{\frac{-1}{2}} - q^m - q^{-m}.$$

Note that the coefficient matrix of equation (35) is of the size $(d+2) \times (d+1)$, while there are only $d+1$ variables α_j to be solved. Matrix B is equal to the upper-left $(M-m+1) \times (M-m)$ submatrix of the square matrix $A - \lambda_m I$, of which the entries a_{ij}, $1 \leqslant i, j \leqslant N$, satisfy the relations $a_{i,j} = (-1)^{i+j} a_{N-j+1,N-i+1}$.

Theorem 3. *For $0 \leqslant m \leqslant M$, the condition of the eigenvalue $\Lambda_m(x) = \lambda_m x^2 + q^m + q^{-m}$ with a non-trivial solution $Q_m(x)$ of equation $(25)_{L=3}$ is determined by the solution of $\det(A - \lambda_m) = 0$, where A is the matrix defined by (36). For each such $\Lambda_m(x)$, there exists a unique (up to constants) non-trivial polynomial solution $Q_m(x)$ of $(25)_{L=3}$, and the degree $Q_m(x)$ is equal to $3M - m$ with $Q_m(0) \neq 0$.*

Proof. By lemma 7, one needs only to show the existence of a non-trivial solution α_j of (35) for a λ_m with $\det(A - \lambda_m) = 0$. For $m = M$, it is obvious as there is no matrix B, and C is zero. For $m < M$, with a given λ_m, there exists a non-trivial vector in the kernel of $A - \lambda_m$,

$$(A - \lambda_m) \begin{pmatrix} \alpha_{3M-m} \\ \vdots \\ \vdots \\ \alpha_{M-m} \end{pmatrix} = \vec{0}.$$

As the u_j appearing in matrix B are all non-zero, by the fact that the rank of B is at most $M - m$, one can extend the above α_j $(M - m \leqslant j \leqslant 3M - m)$ to a solution α_j of (35). The result then follows. □

Remark. By the above theorem, the eigenstate $Q_m(x)$ is unique for a given $\Lambda_m(x)$. It implies that for each m, the eigenvalues $\Lambda_m(x)$ and eigenstates $Q_m(x)$ of the Bethe equation $(25)_{L=3}$

are in one-to-one correspondence. Note that these $Q_m(x)$ are obtained under the constraint of $\Lambda_m(x)$ with form (33), a conclusion by the analysis of the transfer matrix in proposition 1, which we will refer to as the 'physical' criterion while comparing the usual Bethe ansatz technique in the discussion of the next section.

6. The degeneracy and physical solution discussion of the Bethe ansatz relation

In this section, we first discuss the degeneracy relation of eigenspaces of the transform matrix $T(x)$ in $\stackrel{L}{\otimes} \mathbf{C}^{N*}$ with respect to the Bethe solutions we obtained in section 5. As before, $\Lambda(x)$ denotes the eigenvalues of $T(x)$. By proposition 1, the constant term of $T(x)$ is given by

$$T_0 = D + 1 \qquad D := q^{-L} \stackrel{L}{\otimes} Y$$

with the eigenvalue $\Lambda(0)$, which is of the form $q^l + 1$. For $l \in \mathbf{Z}_N$, we denote $\mathbf{E}_L^l =$ the eigensubspace of $\stackrel{L}{\otimes} \mathbf{C}^{N*}$ of the operator D with the eigenvalue q^l, which is of dimension N^{L-1}. By lemma 5, for $0 \leqslant m \leqslant M$, equation (25) describes the relation of $\Lambda(x)$ and $Q_*^+(x)$ through (24) when $\Lambda(0) = q^{2m} + 1$ or $q^{2(N-m)} + 1$. For $L = 1$, $T(x)$ is the constant $q^{-1}Y + I$, and $\mathbf{E}_1^l$ is the eigenspace of Y for the eigenvalue q^{l+1}. By the evaluation at the Baxter vector $|x\rangle_m^+, |x\rangle_{N-m}^+$ respectively, both the spaces, $\mathbf{E}_1^m$ and $\mathbf{E}_1^{N-m}$, give rise to the same functional space generated by $x^m(1 - x^N c^N)B_m(x)$ with $B_m(x)$ in theorem 1. For $L = 2, 3$, expression (13) of T_2 becomes

$$\begin{aligned} L = 2 \qquad & T_2 = q^{-1}c_0c_1(X \otimes Z + Z \otimes X) \\ L = 3 \qquad & T_2 = q^{-2}(c_0c_1 X \otimes Z \otimes Y + c_1c_2 Y \otimes X \otimes Z + c_0c_2 Z \otimes Y \otimes X) \\ & \qquad + q^{-1}(c_0c_1 Z \otimes X \otimes I + c_1c_2 I \otimes Z \otimes X + c_0c_2 X \otimes I \otimes Z). \end{aligned} \tag{37}$$

We shall denote by $\mathcal{O}_L$ the operator algebra generated by the L-tensors of X, Y, Z, I appearing in the corresponding expression of T_2. Then $\mathcal{O}_L$ commutes with D, hence one obtains an $\mathcal{O}_L$-representation on $\mathbf{E}_L^l$ for each l.

$L = 2$. The $\mathcal{O}_2$ is a commutative algebra with the generators $X \otimes Z, Z \otimes X$, and it contains the element $D\ (= ZX \otimes XZ)$. The $\mathcal{O}_2$-representation on $\stackrel{2}{\otimes} \mathbf{C}^{N*}$ has the eigenspace decomposition indexed by the $(X \otimes Z, Z \otimes X)$-eigenvalue (q^i, q^j), or equivalently, the $(D, Z \otimes X)$-eigenvalue (q^l, q^j), where i, j, l are elements in $\mathbf{Z}_N$ with the relation $q^l = q^{j+i}$. In fact, the eigenspace is one dimensional with the basis $\langle\phi_{j_0,j_1}|$ defined by

$$\langle\phi_{j_0,j_1}| := \frac{1}{N^2} \sum_{k,k' \in \mathbf{Z}_N} \omega^{j_0k + j_1k' - kk'} \langle k, k'|$$

where (j_0, j_1) is related to (i, j) by $(\omega^{j_0}, \omega^{j_1}) = (q^i, q^j)$. The vectors $\langle\phi_{j_0,j_1}|$, with $\omega^{j_0+j_1} = q^l$, form a basis of $\mathbf{E}_2^l$. The permutation of tensor factors of $\stackrel{2}{\otimes} \mathbf{C}^{N*}$ induces an automorphism of $\mathbf{E}_2^l$, which interchanges the vectors $\langle\phi_{j_0,j_1}|$ and $\langle\phi_{j_1,j_0}|$. The eigenvalues of T_2 are given by $c_0c_1(q^{l-j-1} + q^{j-1})$ for $j \in \mathbf{Z}_N$, and the corresponding eigenspace is generated by $\langle\phi_{j_0,j_1}|$ and $\langle\phi_{j_1,j_0}|$, where $\omega^{j_1} = q^j, \omega^{j_0+j_1} = q^l$, with the dimension equal to two for all j except $j = (M+1)l$. By (24), the index (l, j) which corresponds to (m, m') of theorem 2 is given by the relation

$$(q^l, q^j) = \left(q^{2m}, q^{m-m'-\frac{1}{2}}\right), \left(q^{-2m}, q^{-m-m'-\frac{1}{2}}\right).$$

With the evaluation at the corresponding Baxter vector, $\langle\phi_{j_0,j_1}|$ and $\langle\phi_{j_1,j_0}|$ give rise to the same eigenstate $B_{m,m'}(x)$ in theorem 2.

$L = 3$. We have

$$qD = (Z \otimes X \otimes I)(X \otimes I \otimes Z)(I \otimes Z \otimes X). \tag{38}$$

With the identification,

$$U = D^{-1/2} Z \otimes X \otimes I \qquad V = D^{-1/2} X \otimes I \otimes Z \tag{39}$$

$\mathcal{O}_3$ is generated by U, V which satisfy the Weyl relation $UV = q^2 VU$ and the Nth power identity. Hence $\mathcal{O}_3$ is the Heisenberg algebra and contains D as a central element. By (37), $qD^{-1/2}T_2$ is the following Hofstadter-like Hamiltonian:

$$c_0c_1(U + U^{-1}) + c_0c_2(V + V^{-1}) + c_1c_2\left(qD^{5/2}UV + q^{-1}D^{-5/2}V^{-1}U^{-1}\right). \tag{40}$$

Note that the above Hamiltonian is a special case of the Faddeev–Kashaev Hamiltonian H_{FK} with $W = q^{-1}D^{-5/2}V^{-1}U^{-1}, \alpha = \beta = \gamma = 1$. Our conclusion on the sector $m = M$ is equivalent to that in [12] as will become clearer later on. It is known that there is a unique (up to equivalence) non-trivial irreducible representation of $\mathcal{O}_3$, denoted by $\mathbf{C}^N_\rho$, which is of dimension N. For each l, $\mathbf{E}^l_3$ is equivalent to N-copies of $\mathbf{C}^N_\rho$ as $\mathcal{O}_3$-modules: $\mathbf{E}^l_3 \simeq N\mathbf{C}^N_\rho$. For $0 \leqslant m \leqslant M$, we consider the space $\mathbf{E}^l_3$ with $q^l = q^{\pm 2m}$. The evaluation of $\mathbf{E}^l_3$ on $|x\rangle^+_{\pm m}$ gives rise to an N-dimensional kernel in $\mathbf{E}^l_3$. By theorem 3, there are N polynomial solutions $Q_m(x)$ of degree $3M - m$ of $(25)_{L=3}$ for each of N distinct eigenvalues $\Lambda_m(x)$. The N-dimensional vector space spanned by those $Q_m(x)$ realizes the irreducible representation $\mathbf{C}^N_\rho$ of the Heisenberg algebra $\mathcal{O}_3$.

Now we discuss the relation between the Bethe equation (25) and the usual Bethe ansatz formulation in the literature. For the physical interest, we focus our attention only on the case of $L = 3$. For $0 \leqslant m \leqslant M$, a solution $Q_m(x)$ in $(25)_{L=3}$ must have $Q_m(0) \neq 0$ by theorem 3. We write

$$\alpha^{-1}_{3M-m} Q_m(x) = \prod_{l=1}^{3M-m}\left(x - \frac{1}{z_l}\right) \qquad z_l \in \mathbf{C}^*. \tag{41}$$

By setting $x = z_l^{-1}$ in $(25)_{L=3}$, we obtain the following relation among z_l, called the Bethe ansatz equation in the physical literature as in the case of usual integrable Hamiltonian chains,

$$q^{m+\frac{3}{2}} \prod_{j=0}^{2} \frac{z_l + c_j}{qz_l - c_j} = \prod_{n=1, n\neq l}^{3M-m} \frac{qz_l - z_n}{z_l - qz_n} \qquad 1 \leqslant l \leqslant 3M - m. \tag{42}$$

The following lemma is obvious.

Lemma 8. *For a polynomial $Q_m(x)$ of form (41), the Bethe ansatz relation (42) for the roots of $Q_m(x)$ is equivalent to the divisibility of $q^{-m}\prod_{j=0}^{2}(1 - xc_jq^{-1})Q_m(xq^{-1}) + q^m\prod_{j=0}^{2}(1 + xc_j)Q_m(xq)$ by $Q_m(x)$. In this situation, the quotient polynomial $\Gamma(x)$ of the latter pair has a degree at most 2 with $\Gamma(0) = q^{-m} + q^m$.*

As mentioned in the remark of theorem 3, the eigenvalue of the Bethe equation $(25)_{L=3}$ is described by the eigenstate $Q_m(x)$, which is a polynomial with the pre-described zeros satisfying (42). However, the quotient polynomial $\Gamma(x)$ in lemma 8 arising from a solution of (42) might not be an eigenvalue $\Lambda_m(x)$ with form (33) of the Bethe equation $(25)_{L=3}$, i.e. $\Gamma(x)$ might not necessarily be an even function. A solution of (42) with a non-even quotient polynomial $\Gamma(x)$ will be called a 'non-physical' one. For the sector $m = M$, there is no non-physical Bethe ansatz solution of (42) by the following lemma.

Lemma 9. *For $m = M$, the quotient polynomial $\Gamma(x)$ in lemma 8 associated with a solution of (42) is always of the form $\Lambda_M(x)$ in (33).*

Proof. By lemma 8, $\Gamma(x) = \sum_{j=0}^{2} \gamma_j x^j$ and $\gamma_0 = q^{-M} + q^M$. It suffices to show the vanishing of γ_1. We write $\alpha_{2M}^{-1} Q_M(x) = x^{2M} + \sum_{j=0}^{2M-1} \beta_j x^j$ with $\beta_0 \neq 0$. By comparing the x-coefficients of $(25)_{L=3}$, one has

$$\gamma_1\beta_0 + \gamma_0\beta_1 = (-q^{-M-1} + q^M)s_1\beta_0 + (q^{-M-1} + q^{M+1})\beta_1$$

which implies $\gamma_1 = 0$. □

With a further argument following the proof of the above lemma, one can obtain relations (5.26) and (5.27)[8] of [12], where the conclusion applies only to the M-sector. In fact, the comparison of the x^2-coefficients of $(25)_{L=3}$ yields the relation

$$\lambda_M = (q^{M-1} + q^M)s_2 + (q^{M+1} - q^{M-1})s_1\beta_1\beta_0^{-1} + (q^{M-1} + q^{M+2} - q^M - q^{M+1})\beta_2\beta_0^{-1}$$

whose equivalent expression is the following:

$$\lambda_M = \left(q^{\frac{-1}{2}} + q^{\frac{-3}{2}}\right)s_2 + \left(q^{\frac{1}{2}} - q^{\frac{-3}{2}}\right)s_1 \sum_{n=1}^{2M} z_n + \left(q^{\frac{3}{2}} + q^{\frac{-3}{2}} - q^{\frac{1}{2}} - q^{\frac{-1}{2}}\right)\sum_{l<n} z_l z_n. \tag{43}$$

With the substitution, $\mu = q^{\frac{1}{2}}c_0^{-1}$, $\nu = q^{\frac{1}{2}}c_1^{-1}$, $\rho = q^{\frac{1}{2}}c_2^{-1}$, expressions $(42)_{m=M}$ and (43) coincide with (5.26) and (5.27) in [12]. By lemma 9, the Bethe ansatz relation (42) is shown to be equivalent to the Bethe equation $(25)_{L=3}$ for the M-sector. However, the parallel statement is not true for the sector $m = M - 1$; in fact, there do exist some 'non-physical' Bethe ansatz solutions to (42). An obvious example is given by the following one. By $3M - m = N$, the collection of inverse of roots of $x^N - \beta = 0$ $(\beta \neq 0)$ forms a solution of the Bethe ansatz equation (42), and its associated quotient polynomial $\Gamma(x)$ in lemma 8 is given by

$$\Gamma(x) = \left(q^{-\frac{3}{2}} + q^{\frac{3}{2}}\right) + \left(q^{-\frac{3}{2}} - q^{\frac{1}{2}}\right)s_1 x + \left(q^{-\frac{3}{2}} + q^{-\frac{1}{2}}\right)s_2 x^2$$

which is not an even polynomial, required by the solutions of equation (25). This shows that the inverse of roots of $x^N - \beta = 0$ for $\beta \neq 0$ provides a 'non-physical' solution of the Bethe ansatz equation (42). By the above example, constraint (33) on the eigenvalue $\Lambda_m(x)$ should be taken into account in the discussion of the Bethe ansatz solutions of (42) for the spectrum problem of the transfer matrix in an arbitrary sector. Such a consideration will become more crucial when the problem of thermodynamic flux limit procedure is involved in the next section.

7. The rational Bethe equation for a generic *q*

In this section, we study the rational Bethe equation for a generic q. In particular for $|q| = 1$, it is the infinity flux limit discussion, i.e. $N \to \infty$, of the models appeared in the last two sections. In the discussion of this section, q will always mean a generic one. By using formula (2), one has the canonical representation on $\mathbf{C}^\infty$ of the Weyl algebra generated by Z, X with the relation

$$ZX = q^2 XZ \qquad Y := ZX.$$

With $\vec{h} \in (\mathbf{P}^3)^L$, one defines the transfer matrix $T_{\vec{h}}(x)$ as in the finite N case, and then discusses its spectrum $\Lambda(x)$. Under the degenerating assumption (14), the method of the previous three sections can be applied equally to reduce the diagonalization problem of the transfer matrix to the Bethe solutions of equation (25). It is important to note that the form of (25) is valid for all finite N while keeping the size L fixed. This enforces us to use the same form of the

[8] The M in our paper is denoted by P in [12].

Bethe equation for a generic q as that in the finite N case, so that the formulation becomes compatible with the infinity N limiting process. An eigenvalue of the transfer matrix, the same for the Bethe equation, should be the generic analogy of $\Lambda(x)$ appeared in the first relation of (24), now denoted by $\widetilde{\Lambda}_m(x) = q^m\Lambda_m(x)$; however, for the eigenstate of the Bethe equation, we will keep the infinity N version of $Q_m(x)$ in (25). The Bethe equation for a generic q for $m \in \mathbf{Z}_{\geqslant 0}$ is now described by

$$\widetilde{\Lambda}_m(x)Q_m(x) = \prod_{j=0}^{L-1}\left(1 - xc_jq^{-1}\right)Q_m(xq^{-1}) + q^{2m}\prod_{j=0}^{L-1}(1 + xc_j)Q_m(xq) \tag{44}$$

where $\widetilde{\Lambda}_m(x)$ is an even polynomial of degree $\leqslant 2[\frac{L}{2}]$ with $\widetilde{\Lambda}_m(0) = q^{2m} + 1$, and $Q_m(x)$ is a formal (power) series of x, i.e.

$$Q_m(x) = \sum_{j\geqslant 0}\alpha_jx^j, \in \mathbf{C}[[x]].$$

As in the discussion of the last two sections, we make a similar analysis on solutions of (44) for $L \leqslant 3$. For $L = 1$, $\widetilde{\Lambda}_m(x) = q^{2m} + 1$, and $B_m(x)$ in theorem 1 defines a formal series, which becomes the basis of the solution space of $(44)_{L=1}$. Note that the description of $\widetilde{\Lambda}_m(x)$ is consistent with the spectrum of the operator Y. For $L = 2, 3$, by expression (37) of T_2, we have the operator algebra $\mathcal{O}_L$ as before, in which T_0 is a central element.

L = 2. Equation $(44)_{L=2}$ is equivalent to the linear systems (29) of α_j, where the coefficients of the equations are defined by (32). By theorem 2, the eigenvalue $\widetilde{\Lambda}_m(x)$ is given by

$$\widetilde{\Lambda}_{m,m'}(x) = \left(q^{m+m'-\frac{1}{2}} + q^{m-m'-\frac{5}{2}}\right)x^2s_2 + q^{2m} + 1 \qquad m' \in \mathbf{Z}_{\geqslant 0}.$$

For $m, m' \in \mathbf{Z}_{\geqslant 0}$, the solutions $Q_m(x)$ of $(44)_{L=2}$ form a one-dimensional space generated by an element $B_{m,m'}(x) \in \mathbf{C}[[x]]$ with $B_{m,m'}(0) = 1$. Note that the algebra $\mathcal{O}_2$ is commutative, and the spectra of the $\mathcal{O}_2$-representation $\overset{2}{\otimes}\ \mathbf{C}^{\infty *}$ give rise to the eigenvalues of $T(x)$, which coincide with the above $\widetilde{\lambda}_{m,m'}(x)$ for $m, m' \in \mathbf{Z}_{\geqslant 0}$.

L = 3. We consider the operator $qD^{-\frac{1}{2}}T_2$, which is the Hamiltonian (40). For the sector m, its eigenvalue is given by $q^{1-m}\widetilde{\lambda}_m$, where $\widetilde{\lambda}_m$ is related to the polynomial $\widetilde{\Lambda}_m(x)$ by

$$\widetilde{\Lambda}_m(x) = \widetilde{\lambda}_mx^2 + q^{2m} + 1.$$

With $\lambda_m := q^{-m}\widetilde{\lambda}_m$, λ_m form the spectra of the quadric-diagonal square matrix A (36) for a generic q, which is now of the infinity size as M increases to ∞. For each $\widetilde{\lambda}_m$, equation $(44)_{L=3}$ of $Q(x)$ is equivalent to system (34). It is easy to see that there exists a unique solution $Q_m(x)$ (up to a constant) by the generic property of q. Note that the same conclusion of $Q_m(x)$ holds equally for an arbitrary complex number $\widetilde{\lambda}_m$, including those not in the spectra of A. This means that solutions of equation $(44)_{L=3}$ alone contain, but not sufficiently determine, the eigenvalues of the transfer matrix $T(x)$. Those $\widetilde{\lambda}_m$ not from the spectra of A correspond to the non-physical Bethe ansatz solutions of the finite N case, as discussed in the previous section.

8. High genus curves and the Hofstadter model

We are now going back to the general situation in section 3. Note that the values ξ_j^N of the curve $\mathcal{C}_{\bar{h}}$ in (8) are determined by ξ_0^N and x^N. Denote

$$y = x^N \qquad \eta = \xi_0^N.$$

The variables (y,η) define the curve

$$\mathcal{B}_{\vec{h}} : C_{\vec{h}}(y)\eta^2 + (A_{\vec{h}}(y) - D_{\vec{h}}(y))\eta - B_{\vec{h}}(y) = 0 \tag{45}$$

where the functions $A_{\vec{h}}, B_{\vec{h}}, C_{\vec{h}}, D_{\vec{h}}$ of y are the following matrix elements:

$$\begin{pmatrix} -A_{\vec{h}}(y) & B_{\vec{h}}(y) \\ C_{\vec{h}}(y) & -D_{\vec{h}}(y) \end{pmatrix} := \prod_{j=0}^{L-1} \begin{pmatrix} -a_j^N & yb_j^N \\ yc_j^N & -d_j^N \end{pmatrix}.$$

Note that $\mathcal{B}_{\vec{h}}$ is a double cover of y-line, and $\mathcal{C}_{\vec{h}}$ is a $(\mathbf{Z}_N)^{L+1}$-branched cover of $\mathcal{B}_{\vec{h}}$. The automorphisms $\tau_\pm$ generate a covering transformation group of $\mathcal{C}_{\vec{h}}$ over $\mathcal{B}_{\vec{h}}$. In this section we shall only consider the case

$$L = 3 \qquad a_0 = d_0 = 0 \qquad b_0 = c_0 = 1$$

and we assume the variables h_1, h_2 to be generic. The expression of $T(x)$ is given by

$$T(x) = x^2(c_1a_2X\otimes Z\otimes Y + a_1b_2Z\otimes Y\otimes X + b_1d_2Z\otimes X\otimes I + d_1c_2X\otimes I\otimes Z)$$

equivalently, $x^{-2}D^{-\frac{1}{2}}T(x)$ is equal to the Hofstadter Hamiltonian (1) with U, V given by (39) and μ,ν,α,β related to h_1, h_2 by

$$\begin{array}{ll} \mu^2 = qb_1c_1a_2d_2 & \alpha^2 = q^{-1}b_1c_1^{-1}a_2^{-1}d_2 \\ \nu^2 = qa_1d_1b_2c_2 & \beta^2 = q^{-1}a_1^{-1}d_1b_2^{-1}c_2. \end{array}$$

The curve $\mathcal{B}_{\vec{h}}$ is defined by (45) with

$$\begin{array}{ll} A_{\vec{h}}(y) = -y^2\left(c_1^Na_2^N + d_1^Nc_2^N\right) & B_{\vec{h}}(y) = y\left(y^2c_1^Nb_2^N + d_1^Nd_2^N\right) \\ C_{\vec{h}}(y) = y\left(y^2b_1^Nc_2^N + a_1^Na_2^N\right) & D_{\vec{h}}(y) = -y^2\left(a_1^Nb_2^N + b_1^Nd_2^N\right). \end{array}$$

By factoring out the y-component, we consider only the main irreducible component of $\mathcal{B}_{\vec{h}}$, denoted by

$$\mathcal{B} : \left(y^2b_1^Nc_2^N + a_1^Na_2^N\right)\eta^2 + \left(a_1^Nb_2^N + b_1^Nd_2^N - c_1^Na_2^N - d_1^Nc_2^N\right)y\eta - \left(y^2c_1^Nb_2^N + d_1^Nd_2^N\right) = 0 \tag{46}$$

which is a double-cover of y-line with four branched points, hence it defines an elliptic curve. For the curve $\mathcal{C}_{\vec{h}}$, the variables ξ_0 and ξ_1 are related by $\xi_0^N = \xi_1^{-N}$. This implies that $\mathcal{C}_{\vec{h}}$ consists of N irreducible components, each one is isomorphic to the same curve $\mathcal{W}$ defined by the equations in the variable $p = (x, \xi_0, \xi_2)$,

$$\mathcal{W} : \xi_0^{-N} = \frac{-\xi_2^Na_1^N + x^Nb_1^N}{x^N\xi_2^Nc_1^N - d_1^N} \qquad \xi_2^N = \frac{-\xi_0^Na_2^N + x^Nb_2^N}{x^N\xi_0^Nc_2^N - d_2^N}.$$

It is easy to see that $\mathcal{W}$ is an N^3-fold (branched) cover of the elliptic curve (46), and the genus of $\mathcal{W}$ is $6N^3-6N^2+1$. We shall label the irreducible components of $\mathcal{C}_{\vec{h}}$ by $s\in\mathbf{Z}_N$ and denote the elements of $\mathcal{C}_{\vec{h}}$ by (p,s) with $p\in\mathcal{W}$, whose ξ_1-coordinate is given by $\xi_0\xi_1 = q^{2s-1}$. Relation (10) now becomes

$$T(x)|p,s\rangle = |\tau_-(p), s-1\rangle q^{2s-1}\Delta_-(p) + |\tau_+(p), s-1\rangle q^{2s}\Delta_+(p)$$

where $\tau_\pm$ are transformations of $\mathcal{W}$ defined by (9), with the coordinates only involving (x,ξ_0,ξ_2), and $\Delta_\pm$ are the following functions on $\mathcal{W}$:

$$\Delta_-(p) = -x\xi_0^{-1}(x\xi_2c_1 - d_1)(x\xi_0c_2 - d_2)$$

$$\Delta_+(p) = x\xi_2\frac{(a_1d_1 - x^2b_1c_1)(a_2d_2 - x^2b_2c_2)}{(\xi_2a_1 - xb_1)(\xi_0a_2 - xb_2)} \qquad p = (x,\xi_0,\xi_2)\in\mathcal{W}.$$

By averaging the vectors $|p, s\rangle$ over an element p of $\mathcal{W}$, one defines the following Baxter vector on $\mathcal{W}$:

$$|p\rangle := \frac{1}{N}\sum_{s=0}^{N-1}|p, s\rangle q^{s^2}.$$

The action of the transform matrix $T(x)$ on $|p, s\rangle$ can be descended to the Baxter vector of $\mathcal{W}$ as follows:

$$x^{-2}T(x)|p\rangle = |\tau_-(p)\rangle\widetilde{\Delta}_-(p) + |\tau_+(p)\rangle\widetilde{\Delta}_+(p) \tag{47}$$

where $\widetilde{\Delta}_\pm$ are the functions of $\mathcal{W}$,

$$\widetilde{\Delta}_-(x, \xi_0, \xi_2) = \frac{(x\xi_2 c_1 - d_1)(x\xi_0 c_2 - d_2)}{-x\xi_0}$$
$$\widetilde{\Delta}_+(x, \xi_0, \xi_2) = \frac{\xi_2(a_1 d_1 - x^2 b_1 c_1)(a_2 d_2 - x^2 b_2 c_2)}{x(\xi_2 a_1 - x b_1)(\xi_0 a_2 - x b_2)}.$$

By the following component expression of the Baxter vector on $\mathcal{C}_{\vec{h}}$,

$$\langle k_0, k_1, k_2|p, s\rangle q^{s^2} = q^{(s-k_0-k_1)^2} q^{-2k_0^2-2k_0k_1-k_1^2+k_1} \prod_{i=1}^{k_1}\frac{\xi_0(-\xi_2 a_1\omega^i + x b_1)}{\xi_2 x c_1\omega^i - d_1}\prod_{j=1}^{k_2}\frac{-\xi_0 a_2\omega^j + x b_2}{\xi_2(\xi_0 x c_2\omega^j - d_2)}$$

the Baxter vector on $\mathcal{W}$ is given by

$$\langle k_0, k_1, k_2|p\rangle = \frac{\sum_{n=0}^{N-1} q^{n^2}}{N} q^{-2k_0^2-2k_0k_1-k_1^2+k_1} \prod_{i=1}^{k_1}\frac{\xi_0(-\xi_2 a_1\omega^i + x b_1)}{\xi_2 x c_1\omega^i - d_1}\prod_{j=1}^{k_2}\frac{-\xi_0 a_2\omega^j + x b_2}{\xi_2(\xi_0 x c_2\omega^j - d_2)}. \tag{48}$$

Note that k_j in the above formula are integers modular N. Each product term on the right-hand side means the one for a positive integer k_j representing its class in $\mathbf{Z}_N$. For an eigenvector $\langle\varphi|$ in $\overset{3}{\otimes}\,\mathbf{C}^{N*}$ of the operator $x^{-2}T(x)$ with the eigenvalue λ, by (47), the function $Q(p) := \langle\varphi|p\rangle$ of $\mathcal{W}$ satisfies the following Bethe equation:

$$\lambda Q(p) = Q(\tau_-(p))\widetilde{\Delta}_-(p) + Q(\tau_+(p))\widetilde{\Delta}_+(p) \qquad \lambda \in \mathbf{C}. \tag{49}$$

The above equation possesses a $\mathbf{Z}_2$-symmetry with respect to the following involution of $\mathcal{W}$:

$$\sigma : \mathcal{W} \longrightarrow \mathcal{W} \qquad p = (x, \xi_0, \xi_2) \mapsto \sigma(p) = (-x, -\xi_0, -\xi_2).$$

In fact, the commutativity of σ and $\tau_\pm$, and the σ-invariant property of $\widetilde{\Delta}_\pm(p)$ are easily seen. Then by (48), the Baxter vector $|p\rangle$ is invariant under σ, i.e. $|p\rangle = |\sigma(p)\rangle$ for $p \in \mathcal{W}$, which implies that $Q(p)$ is a σ-invariant function. Furthermore, the rational function $Q(p)$ has the poles contained in the following divisor:

$$\xi^{N-1}\prod_{i=1}^{N-1}(\xi_2 x c_1\omega^i - d_1)(\xi_0 x c_2\omega^i - d_2) = 0.$$

In particular, it is regular at $x = 0, \infty$. Note that the finite values of $Q(p)$ at $x = 0, \infty$ are consistent with the asymptotic values of $\widetilde{\Delta}_\pm$ at $x = 0, \infty$ in equation (49),

$$\widetilde{\Delta}_\pm(x, \xi_0, \xi_2) = \pm x^{-1}\xi_0^{-1}d_1 d_2 + O(1) \qquad \text{as} \quad x \to 0$$
$$\widetilde{\Delta}_\pm(x, \xi_0, \xi_2) = \pm x\xi_2 c_1 c_2 + O(1) \qquad \text{as} \quad x \to \infty.$$

With the x-coordinate, $\mathcal{W}$ is a $2N^2$-cover over the x-line, unramified at points over $x = 0, \infty$ whose (x, ξ_0, ξ_2)-coordinates are given by

$$\begin{aligned} 0^{\pm}_{i,i'} &= \pm \left(0, q^i \sqrt{\frac{d_1 d_2}{a_1 a_2}}, q^{i'} \sqrt{\frac{d_1 a_2}{a_1 d_2}}\right) \\ \infty^{\pm}_{i,i'} &= \pm \left(\infty, q^i \sqrt{\frac{c_1 c_2}{b_1 b_2}}, q^{i'} \sqrt{\frac{c_1 b_2}{b_1 c_2}}\right) \qquad i, i' \in \mathbf{Z}_N. \end{aligned} \tag{50}$$

Consider the D-eigenspace decomposition of $\overset{3}{\otimes}\, \mathbf{C}^{N*} = \bigoplus_{l \in \mathbf{Z}_N} \mathbf{E}^l_3$ in section 6. The evaluation on the Baxter vector of $\mathcal{W}$ gives rise to the following linear transformation:

$$\varepsilon_l : \mathbf{E}^l_3 \longrightarrow \{\text{rational functions of } \mathcal{W}\} \qquad v \mapsto \varepsilon_l(v)(p) := \langle v | p \rangle \qquad \text{for} \quad l \in \mathbf{Z}_N.$$

Theorem 4. *For $l \in \mathbf{Z}_N$, the linear map ε_l is injective, hence it induces an identification of $\mathbf{E}^l_3$ with an N^2-dimensional functional space of $\mathcal{W}$.*

Proof. Define the following vectors in $\overset{3}{\otimes}\, \mathbf{C}^{N*}$:

$$\begin{aligned} \langle \psi_{k_0,k_1,k_2} | &= \left(\sum_{n=0}^{N-1} q^{n^2} \right)^{-1} \sum_{k' \in \mathbf{Z}_N} \omega^{k_1 k'} \langle k_0, k', k_2 | \\ \langle \phi_{j_0,j_1,j_2} | &= \sum_{k \in \mathbf{Z}_N} q^{2k(-j_0+j_1+j_2)-k(k-1)} \langle \psi_{j_1-k,k,j_2-k} | \end{aligned}$$

where $k_i, j_i \in \mathbf{Z}_N$. We have

$$\begin{aligned} \langle \psi_{k_0,k_1,k_2} | Z \otimes X \otimes I &= \omega^{k_0+k_1} \langle \psi_{k_0,k_1,k_2} | \\ \langle \psi_{k_0,k_1,k_2} | X \otimes I \otimes Z &= \omega^{k_2} \langle \psi_{k_0-1,k_1,k_2} | \\ \langle \psi_{k_0,k_1,k_2} | I \otimes Z \otimes X &= \langle \psi_{k_0,k_1+1,k_2-1} | . \end{aligned}$$

By (38), one has

$$\langle \psi_{k_0,k_1,k_2} | D = q^{-1+2(k_0+k_1+k_2)} \langle \psi_{k_0-1,k_1+1,k_2-1} | \qquad \langle \phi_{j_0,j_1,j_2} | D = q^{-1+2j_0} \langle \phi_{j_0,j_1,j_2} | .$$

For a given $l \in \mathbf{Z}_N$, let j_0 be the element in $\mathbf{Z}_N$ defined by $q^l = q^{-1+2j_0}$. Then the vectors $\langle \phi_{j_0,j_1,j_2} |$ with $j_1, j_2 \in \mathbf{Z}_N$ form a basis of $\mathbf{E}^l_3$. By (48), we have

$$\langle \psi_{k_0,k_1,k_2} | p \rangle = N^{-1} q^{-2k_0^2} \sum_{k' \in \mathbf{Z}_N} q^{-k'^2+(2k_1-2k_0+1)k'} \prod_{i=1}^{k'} \frac{\xi_0(-\xi_2 a_1 \omega^i + x b_1)}{\xi_2 x c_1 \omega^i - d_1} \prod_{j=1}^{k_2} \frac{-\xi_0 a_2 \omega^j + x b_2}{\xi_2(\xi_0 x c_2 \omega^j - d_2)}$$

hence

$$\begin{aligned} \langle \phi_{j_0,j_1,j_2} | p \rangle = N^{-1} q^{-2j_1^2} & \sum_{k,k' \in \mathbf{Z}_N} q^{-3k^2+k(-2j_0+6j_1+2j_2+1)-k'^2+4k'k+(-2j_1+1)k'} \\ & \times \prod_{i=1}^{k'} \frac{\xi_0(-\xi_2 a_1 \omega^i + x b_1)}{\xi_2 x c_1 \omega^i - d_1} \prod_{j=1}^{j_2-k} \frac{-\xi_0 a_2 \omega^j + x b_2}{\xi_2(\xi_0 x c_2 \omega^j - d_2)}. \end{aligned}$$

Set $p = 0^{\pm}_{i,i'}$ defined in (50). By the relations of their (ξ_0, ξ_2)-coordinates, $\xi_0 \xi_2 a_1 d_1^{-1} = q^{i+i'}$, $\xi_0 \xi_2^{-1} a_2 d_2^{-1} = q^{i-i'}$, we obtain

$$\begin{aligned} \langle \phi_{j_0,j_1,j_2} | 0^{\pm}_{i,i'} \rangle &= N^{-1} q^{-2j_1^2+j_2^2+j_2(1+i-i')} \sum_{k,k' \in \mathbf{Z}_N} q^{-2k^2+k(-2j_0+6j_1-i+i')} q^{k'(-2j_1+2+i+i'+4k)} \\ &= q^{j_0(1-j_1)-2j_1+\frac{j_1^2}{2}+j_2+j_2^2+\frac{-j_1(3i+i')+2j_2(i-i')}{2}+\frac{j_0(i+i')}{2}-\frac{(i+i'+2)(-i+3i'+2)}{8}} \end{aligned}$$

hence

$$q^{j_0(-1+j_1)+2j_1-\frac{j_1^2}{2}-j_2-j_2^2}\left\langle \phi_{j_0,j_1,j_2} \middle| 0^{\pm}_{i,i'} \right\rangle q^{\frac{(i+i'+2)(-i+3i'+2)-4j_0(i+i')}{8}} = q^{\frac{i(-3j_1+2j_2)-i'(j_1+2j_2)}{2}}. \tag{51}$$

As the correspondence, $(j_1, j_2) \mapsto (-3j_1+2j_2, -j_1-2j_2)$, defines an automorphism of $\mathbf{Z}_N^2$, relation (51) gives rise to an isomorphism between $\mathbf{E}_l$ and the space of Baxter vectors $0^+_{i,i'}$ (or equivalently $0^-_{i,i'}$). This implies the injectivity of ε_l. □

From the discussion in section 6, $\mathbf{E}_3^l$ is equivalent to N copies of the standard representation as the Heisenberg algebra $\mathcal{O}_3$-modules. Hence by theorem 4, there exists an $\mathcal{O}_3$-module structure on $\varepsilon_l\big(\mathbf{E}_3^l\big)$, inherited from the representation space $\mathbf{E}_3^l$. The mathematical structure of the functional space $\varepsilon_l\big(\mathbf{E}_3^l\big)$ by incorporating the divisor theory of Riemann surfaces into the Heisenberg algebra representation remains an algebraic geometry problem for further study.

9. Conclusions and perspectives

We follow the framework in [12] by the quantum integrable method to study the diagonalization problem of some Hofstadter-like models. Through the Baxter vector of the spectral curve, the study of diagonalizing a Hamiltonian with a rational magnetic flux is reduced to the problem of a certain 'Strum–Liouville-like' difference equation on the curve, called the Bethe equation of the associated model. The spectral curve has in general, a large genus, and the relations among zeros and poles for a solution of the Bethe equation yield a system of algebraic equations. Such systems of relations among zeros of the Bethe polynomial solution on a rational spectral curve are usually referred to the Bethe ansatz equation in the literature. For certain models of physical interest, e.g., the Hofstadter Hamiltonian (1), the study of the high genus spectral curve is a necessary step in solving the spectrum problem through the algebraic Bethe ansatz technique. In this paper, we clarify some finer mathematical manipulations in [12], then go through all the delicate points one must consider in order to obtain the explicit Bethe solutions. A careful analysis of the mathematical nature of the Bethe equation reveals the vital role of algebraic geometry in a thorough understanding of the Bethe ansatz method, as is the need to obtain the physical answer of the associated model. For this reason, we have examined, in this paper, the Bethe ansatz equation in the context of algebraic geometry, even in the degenerated rational spectral curve case in order to gain mathematical insight into the Bethe equation. We further extend the approach to some more general situations. Above all, we have endeavoured to present a clear and self-contained account of the theory, and hope to have elucidated the mathematical structure of the Bethe-ansatz-style method in the physical literature. The main content of this paper is in the discussions after section 5, where the detailed mathematical derivation and analysis comparable to physical considerations are presented. The topics between sections 4 and 7 are devoted to the degenerated case, where the Bethe equations are related to models with the rational spectral curve. With an explicit gauge choice, we have conducted the mathematical investigation of the Bethe equation and obtained the complete Bethe solutions for all sectors in section 5. With these mathematical results, we are able to further advance the study of the relevant physical problems, namely, the 'degeneracy' of eigenstates of the transfer matrix to the Bethe solutions and the thermodynamic limit discussion in sections 6 and 7. Furthermore in section 6, the explicit calculation we have performed for the Bethe solutions, when specializing on one particular sector, provides results parallel to those using the usual Bethe ansatz method in [12]. Meanwhile, the finding of some non-physical Bethe ansatz solutions in other sectors supports the justification for our approach to the problem. The method we employ here can be applied equally well to any number of size L. However, to keep things simple, we restrict our attention in this paper only to the case $L \leqslant 3$ in the discussion of Bethe solutions; the analyses on $L = 1, 2$ are made mainly for mathematical purpose to pave the way for discussing the models with a higher size L. The Hofstadter-like model is related to $L = 3$, of which the Bethe ansatz equation has been mathematically discussed in detail here. For $L = 4$, it is expected that the problem would be closely related to the discrete quantum pendulum by the works of [9, 18]. The results we have obtained in this paper through the Bethe equation approach strongly indicate a promising direction to the spectra problem of other models, e.g. the discrete quantum pendulum. For the thermodynamic limit discussion, the analysis in section 7 on the special Hofstadter-like Hamiltonian (40) for a generic q shows that the Bethe relation on the spectrum and eigenstates we proposed are in accordance with the diophantine approximation process of an irrational flux. The comparison of our Bethe ansatz method with the C^*-algebra approach of semiclassical analysis employed in [6] for the multifractal spectrum structure is a fascinating problem. We plan to address the question of such a program elsewhere.

For the original Hofstadter model (1), the Bethe equation is formulated as a 'difference' equation of functions on a high genus spectral curve. In section 8, we have made a primary investigation on its solutions. As the spectral curve is a Galois Abelian cover of an elliptic curve with the covering group determined by the order N of the rational flux, it would be essential to have a detailed algebraic geometry study of such a high genus curve in accordance with the Bethe solutions, so that the base elliptic curve and the classical elliptic functions could be engaged in the theory. With our finding in the rational spectral curve case as guidance for some appropriate direction of calculation, the analysis we have made in this paper will serve as a basis for further study of the Hofstadter model through the elliptic curve techniques in algebraic geometry. This approach would allow us to follow a similar path as in the rational spectral curve case for the study of the spectrum problem of the Hofstadter model. This rich structure requires further study, and a scheme along this line of interpretation is under current investigation. Indeed, we hope that our efforts would eventually shed new light on the role of algebraic geometry in exactly solvable integrable models. In this paper, we restrict our attention only to certain Hofstadter-like models, and leave possible generalizations and applications to future work.

Acknowledgments

One of the authors, S-S Roan, would like to thank R Seiler and J Kellendonk for the stimulating conversations on the Hofstadter model during their encounters in U.C. Berkeley, USA, in the spring of 1998. He would acknowledge informative discussions and communications with B M McCoy on R Baxter's work. We would also like to thank M Jimbo for inspiring discussions during the final stage of this paper. This work was reported by the second author in the workshop 'Quantum Algebra and Integrabilty' CRM Montreal, Canada, 2–15 April 2000, and as part of the subject of an Invited Lecture at APCTP-Nankai Joint Symposium on 'Lattice Statistics and Mathematical Physics', Tianjin, China, 7–11 October 2001, for which he would like to thank them for their invitation and hospitality.

References

[1] Azbel M Ya 1964 Energy spectrum of a conduction electron in a magnetic field *Sov. Phys.–JETP* **19** 634–45

[2] Abanov A G, Talstra J C and Wiegmann P B 1998 Hierarchical structure of Azbel–Hofstadter problem: strings and loose ends of Bethe ansatz *Nucl. Phys.* B **525** 571–96 (*Preprint* cond-mat/9711274)

[3] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)

[4] Baxter R J, Bazhanov V V and Perk J H H 1990 Functional relations for transfer matrices of the chiral Potts model *Int. J. Mod. Phys.* B **4** 803–70

[5] Bazhanov V V and Stroganov Yu G 1990 Chiral Potts model as a descendant of the six-vertex model *J. Stat. Phys.* **59** 799–817

[6] Bellissard J 1994 Non commutative methods in semiclassical analysis *Lecture Notes in Mathematics* vol 1589 (Berlin: Springer) pp 1–64

[7] Bellissard J and Simon B 1982 Cantor spectrum for the almost Mathieu equation *J. Funct. Anal.* **48** 408–19

[8] Buslaev V and Fedotov A 1994 *The Functional Structure of the Monodromy Matrix for Harper's Equation* vol 70 (Basel: Birkhäuser)

[9] Bobenko A, Kutz N and Pinkall U 1993 The discrete quantum pendulum *Phys. Lett.* A **177** 399–404

[10] Chambers W G 1965 Linear network model for magnetic breakdown in two dimensions *Phys. Rev.* A **140** 135–43

[11] Choi M, Elliott G A and Yui N 1990 Gauss polynomials and rotational algebra *Invent. Math.* **99** 225–46

[12] Faddeev L D and Kashaev R M 1995 Generalized Bethe ansatz equations for Hofstadter problem *Commun. Math. Phys.* **155** 181–91 (*Preprint* hep-th/9312133)

[13] Floratos E G and Nicolis S 1998 An *SU*(2) analog of the Azbel–Hofstadter Hamiltonian *J. Phys. A: Math. Gen.* **31** 3961–75 (*Preprint* hep-th/9508111)

[14] Harper P G 1955 Single band motion of conduction electrons in a uniform magnetic field *Proc. Phys. Soc. London* A **68** 874–92

[15] Helffer B and Sjöstrand J 1988 Analyse semi-classique pour l'equation de Harper I avec application à l'étude de l'équation de Schrödinger avec champ magnétique *Bulletin de la Société Mathématique de France* 116 Fasc. 4, Mémoire 34 1–113

Helffer B and Sjöstrand J 1990 Analyse semi-classique pour l'equation de Harper II *Bulletin de la Société Mathématique de France* 118 Fasc. 1, Mémoire 40 1–139

Helffer B and Sjöstrand J 1989 Semi-classical analysis for Harper's equation III, Cantor structure of the spectrum *Memoire de la Société Mathématique de France* 117 Fasc. 4 1–124

[16] Hofstadter D R 1976 Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields *Phys. Rev.* B **14** 2239–49

[17] Kreft Ch and Seiler R 1996 Models of the Hofstadter type *J. Math. Phys.* **37** 5207–43

[18] Kellendonk J, Kutz N and Seiler R 1999 Spectra of quantum integrals *Discrete Integrable Geometry and Physics (Oxford Lectures Series in Mathematics and Applications vol 16)* ed A I Bobenko and R Seiler (New York: Oxford University Press) pp 247–97

[19] Langbein D 1969 The tight-binding and the nearly-free-electron approach to lattice electrons in the external magnetic fields *Phys. Rev.* **180** 633–48

[20] Peierls R 1933 Zur theorie des diamagnetismus von leitungselektronen *Z. Phys.* **80** 763–91

[21] Shubin M A 1994 Discrete magnetic Laplacian *Commun. Math. Phys.* **164** 259–75

[22] Takhtadzhan L A and Faddeev L D 1979 The quantum method of the inverse problem and the Heisenberg *XYZ* model *Russ. Math. Surv.* **34** 11–68

[23] Wiegmann P B and Zabrodin A V 1994 Bethe-ansatz approach to Hofstadter problem, quantum group and magnetic translations *Nucl. Phys.* B **422** 495–514

Wiegmann P B and Zabrodin A V 1995 Algebraization of difference eigenvalue equations related to $U_q(sl_2)$ *Nucl. Phys.* B **451** 699–724 (*Preprint* cond-mat/9501129)

[24] Wilkinson M 1984 Critical properties of electron eigenstates in incommensurate systems *Proc. R. Soc.* A **391** 305–30

[25] Zak J 1964 Magnetic translation group *Phys. Rev.* A **134** 1602–07

Zak J 1964 Magnetic translation group: II. Irreducible representations *Phys. Rev.* A **134** 1607–11